Computationally Efficient Codes for Adversarial Binary-Erasure Channels

Short informal talk @ Imperial College

Anand D. Sarwate (Rutgers University)

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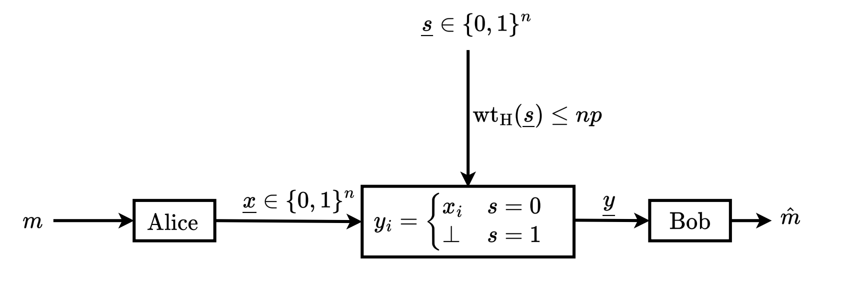
Joint work with Sijie Li (UT Austin), Prasad Krishnan (IIT Hyderabad), Sidharth Jaggi (U. Bristol), Michael Langberg (U. at Buffalo)

Our good friend the erasure channel



The erasure channel almost needs no introduction...

- Binary input, ternary output.
- Think of erasures as a state sequence $\underline{s}$ where 1 means "erase."
- Fraction of erased bits upper bounded by p, either exactly (coding theory) or with high probability (Shannon theory).

Optimism versus pessimism

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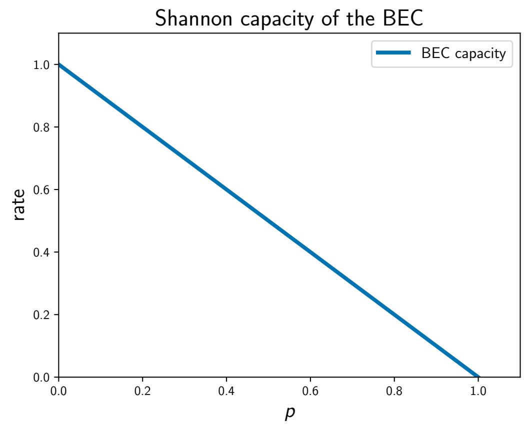

Optimistic: *Random* erasures (BEC): erasures are i.i.d. Bernoulli. Shannon capacity is $\mathbf{1 - p}$.

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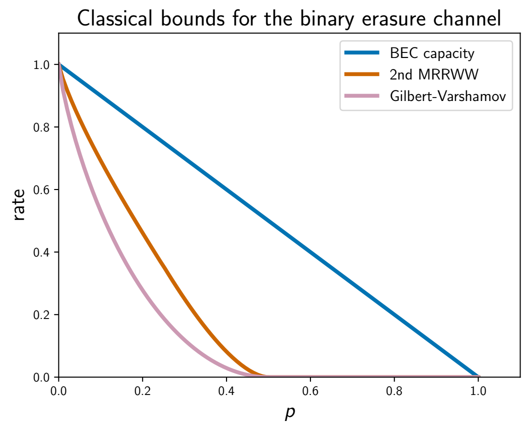
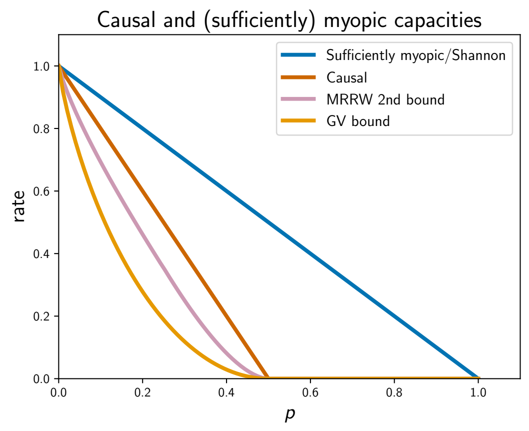


Optimistic: *Random* erasures (BEC): erasures are i.i.d. Bernoulli. Shannon capacity is $\mathbf{1 - p}$.

Pessimistic: *Adversarial* erasures: erasures can depend on transmitted codeword. **Capacity unknown**!

- Lower bound: Gilbert-Varshamov (and **linear codes** work.)
- Upper bound from linear programming (MRRW "LP" bound).

Models in the middle: causal and myopic



(Chen, Jaggi, Langberg 2015) (Sarwate 2010) (Dey et al. 2016) (Dey et al. 2019)

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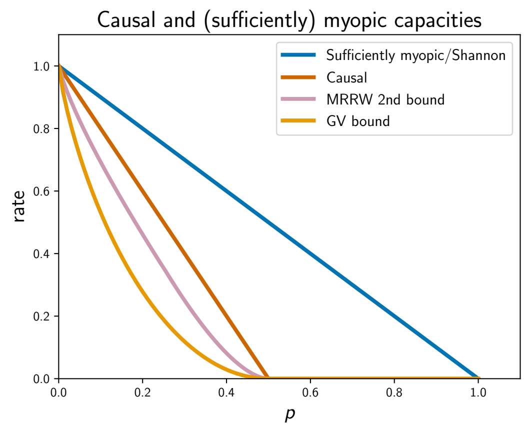

Causal adversaries: erasures can depend on the current and past input only.

Capacity is $\mathbf{1-2p}$.

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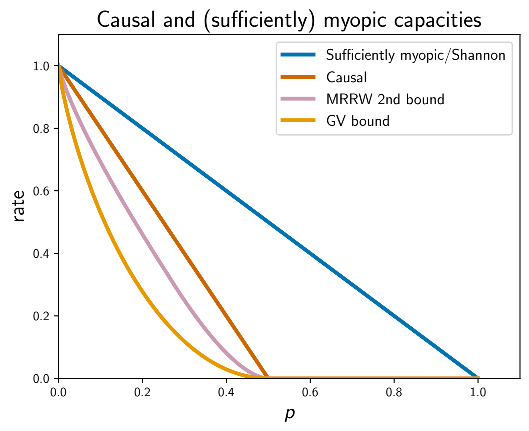

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Myopic adversaries: erasures can depend on a noisy (erased by BEC(q)) version of the codeword.

- If $p < q$, capacity is $\mathbf{1 - p}$.
- If $p > q$... see the paper.

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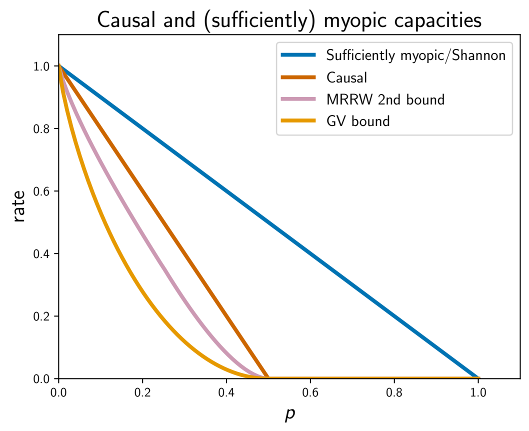

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Achievability arguments use **stochastic encoding** and **list decoding** with **nonlinear codes**.

(Chen, Jaggi, Langberg 2015) (Sarwate 2010) (Dey et al. 2016) (Dey et al. 2019)

Main questions we address

Can we design **efficient codes** for causal and myopic models?

By **efficient** we mean that they take **polynomial time** to encode, decode, and store.

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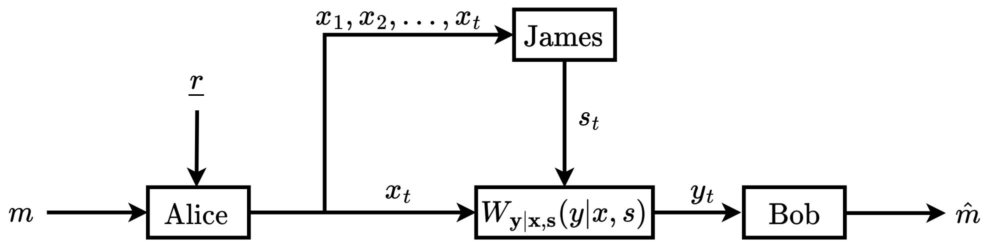
  $\longrightarrow$ use a **library of linear codebooks**.
- **common randomness** is unrealistic.
  $\longrightarrow$ use **limited encoder randomization** to **confuse the adversary**.
- **minimum distance coding** is not efficient in general.
  $\longrightarrow$ use **list decoding** to permit **efficient decoding**.

Causal and Myopic Adversaries

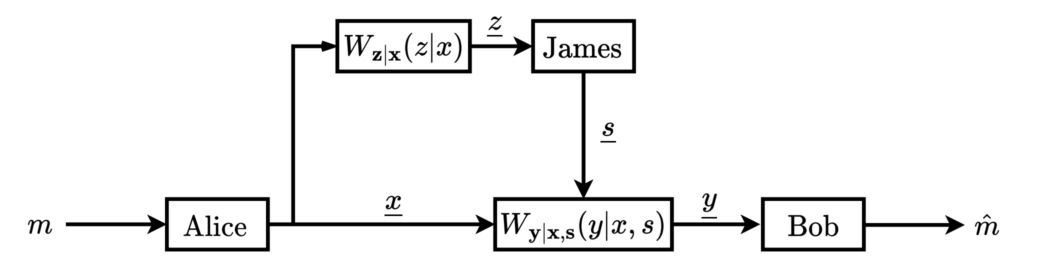
Causal/Online Adversaries



A **causal adversary** can eavesdrop **noiselessly and in real time** on the channel inputs:

- Decision on whether to erase at time t can depend on $(x_1, x_2, \ldots, x_t)$.
- **Adversary's budget:** at most np erasures in codeword of length n.

Myopic adversaries



A **myopic adversary** can eavesdrop **noisily and noncausally** on the channel inputs:

- Decision on whether to erase at time t can depend on $\underline{z}$ formed by passing $\underline{x}$ through a BEC with erasure probability q
- **Adversary's budget:** at most np erasures in codeword of length n.

"Efficient" coding schemes

To get **polynomial complexity**, use

- **a small amount of randomization** to select from a
- **library of random linear codes** and
- uses **list decoding** to reduce the search space

There are different types of complexity we would like to control:

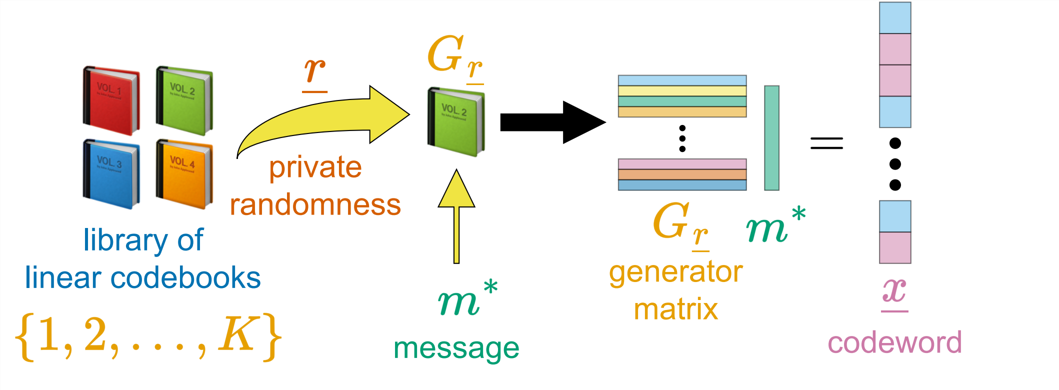
- **Design**: how many bits do we need to generate the code?
- **Storage**: how many bits do we need to store the code?
- **Encoding**: how many operations are needed to encode a message?
- **Decoding**: how many operations are needed to decode the message?

Main results

Model **rate**	**Randomness**	**Enc/Storage**	**Decoding**	$\mathbf{P}_{\text{error}}$
Myopic $p < q$ $\mathbf{1 - p - \epsilon}$	$\lambda_{SM} \log(n)$	$O(n^{2+\lambda_{SM}})$	$O(n^{3+\lambda_{SM}})$	$O(n^{-\lambda_{SM}})$
Myopic $q < p$ **small rate**	$O(n \log \log n)$	$O(n^2 \log \log n)$	$O(n^3 \log \log n)$	$O(n^{-4/5})$
Causal $\mathbf{1 - 2p - \epsilon}$	$O\left(\frac{\gamma \log n}{\epsilon}\right)$	$O(n^3 \log \log n)$	$O(n^{32/\epsilon})$	$O(n^{-(\gamma-1)})$

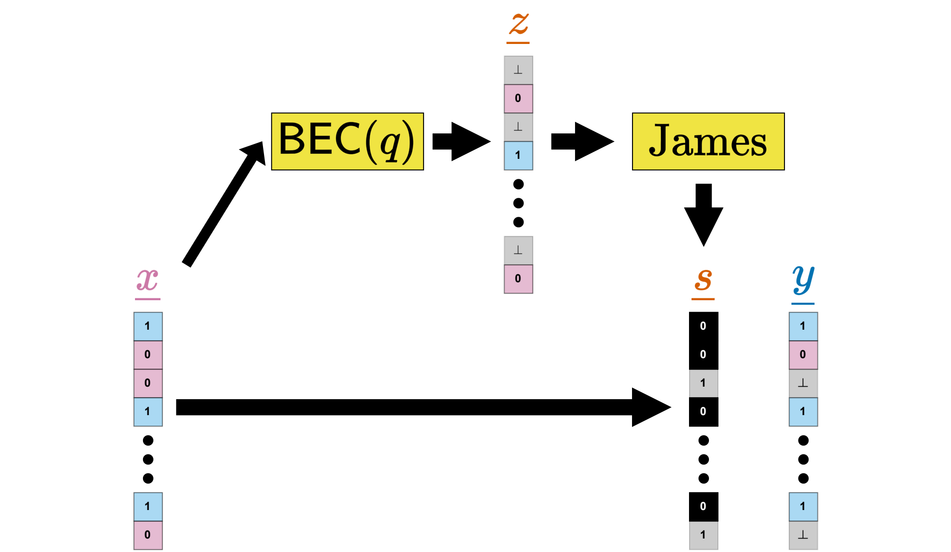
Sufficiently myopic adversaries

Encoding uses a library of linear codebooks

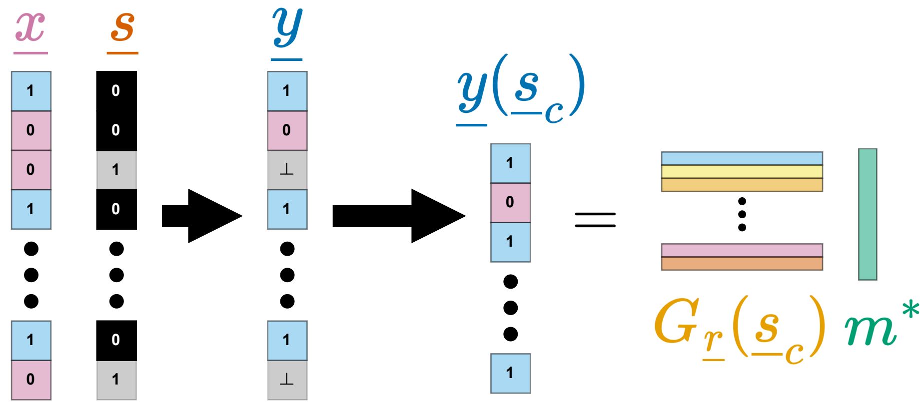


Generating random linear codes: $K = 2^{n\lambda_{SM}}$ generator matrices $G_i \in \mathbb{F}_2^{n \times nR}$ generated i.i.d. $\mathrm{Bernoulli}(1/2)$.

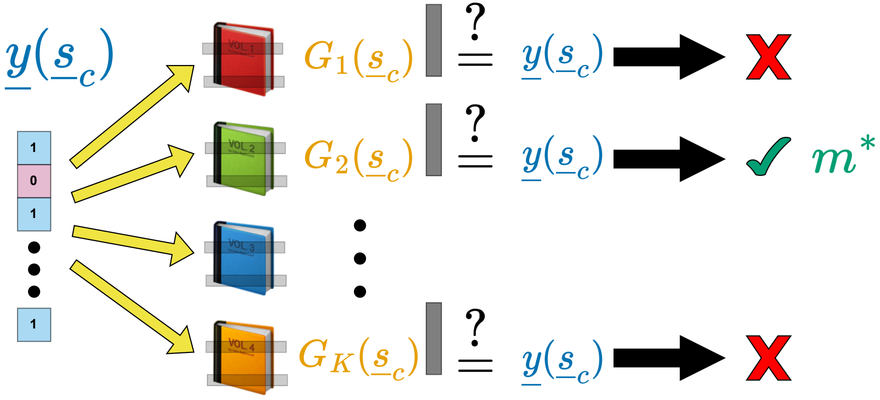
James sees an erased version of the codeword



Look at “unerased” rows of codebook

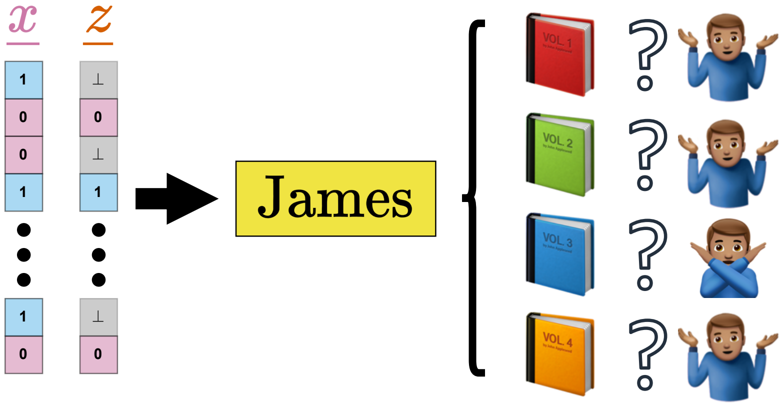


Decoder just tries every codebook



Complexity: n^3 per codebook, $K = n^{n\lambda_{SM}}$ codebooks.

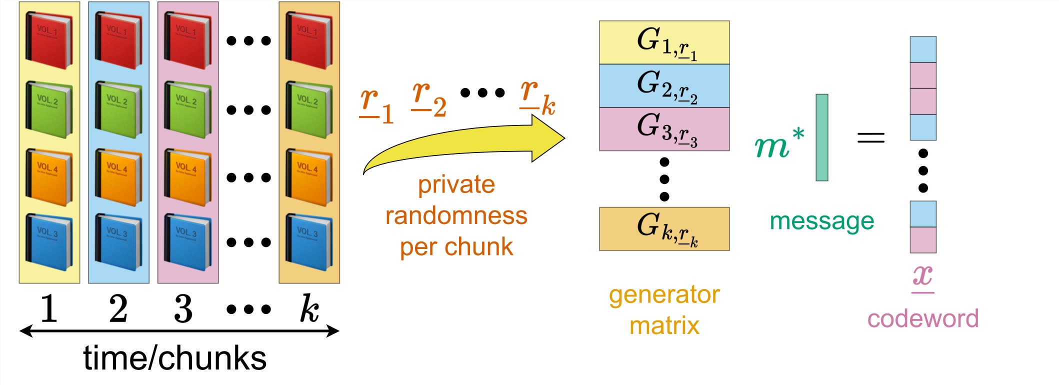
Myopia helps: if $q > p$, James cannot guess the correct codebook



$$P_{\text{error}} = O\left(\mathbf{n}^{-\lambda_{\mathsf{SM}}}\right)$$

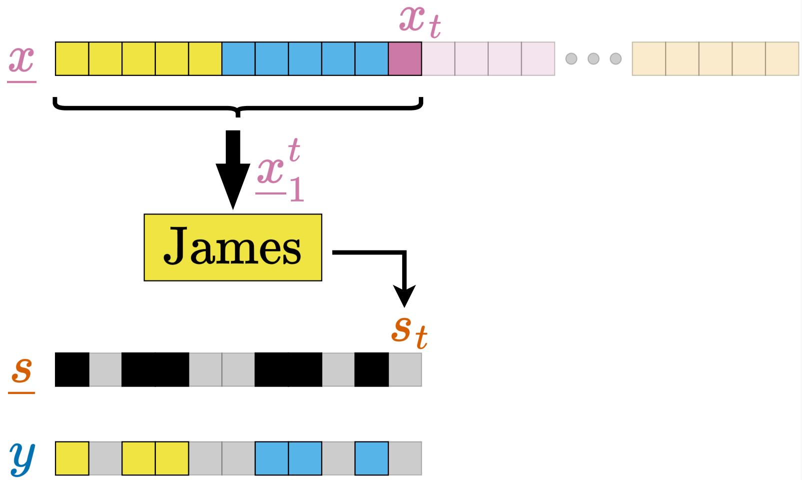
Causal adversaries

Encode splits block into a constant $k = \lceil \frac{n}{\epsilon} \rceil$ chunks

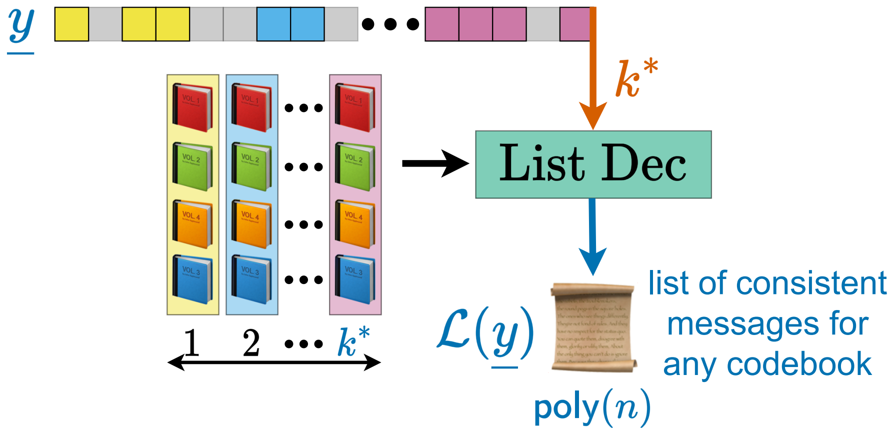


Generate a library of linear codebooks independently for each chunk.

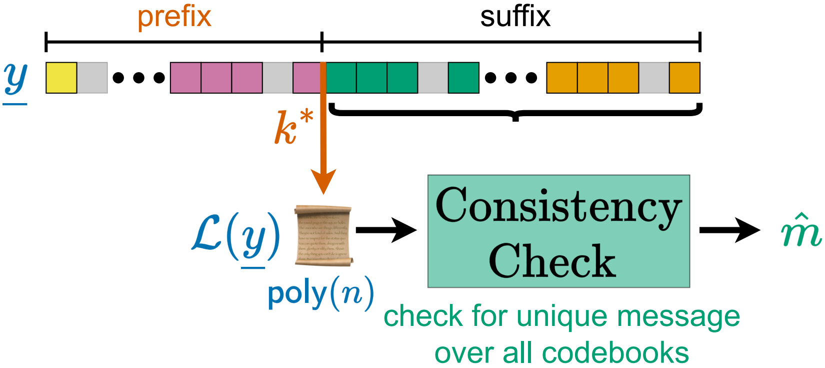
James can erase with causal information only



Bob decodes to a polynomial list after a certain time



Bob uses suffix to disambiguate the list



Why does this work?

1. Bob can **track James's erasure budget**.
2. List decoding creates **a smaller set of messages** to check for consistency.
3. James has a choice to **make the list larger** (erase more earlier, less later) or **conserve his budget** (erase less earlier, more later).
4. **Poor James, he can't win.**

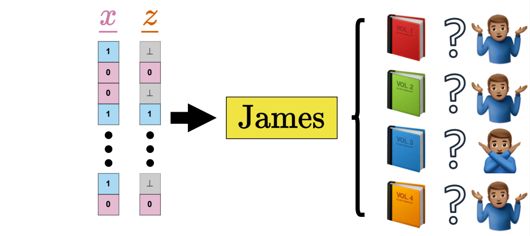
Recap and next steps

Recap

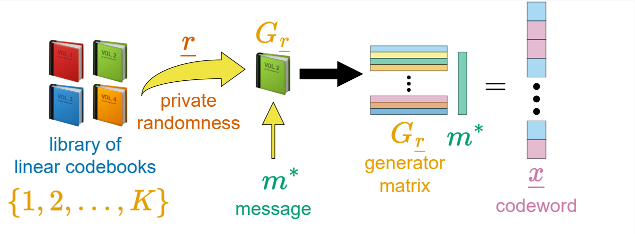
We design efficient (polynomial time) codes for both causal and myopic models.

- Use **libraries of linear codebooks** for **efficient decoding**.
- Use **limited encoder randomization** to **confuse the adversary**.
- Use **list decoding** to permit **efficient decoding**.

Open questions and future directions



- Other adversary structres?
- Better degree for "polynomial"?
- Better error guarantees?
- General AVC models?



$\underline{r}$
private randomness
$G_{\underline{r}}$
library of linear codebooks
$\{1, 2, \ldots, K\}$
m^*
message
$G_{\underline{r}}$
generator matrix
m^*
$=$
$\underline{x}$
codeword

Thank you!