Flexible Tensor Decompositions for Learning and Optimization

Anand D. Sarwate, Rutgers University

28 March 2025

Center for Information and Systems Engineering
Boston University

Tensors: what are they good for?

The history of the word “tensor”

Let’s meet some 19th century physicists

All images: Wikipedia

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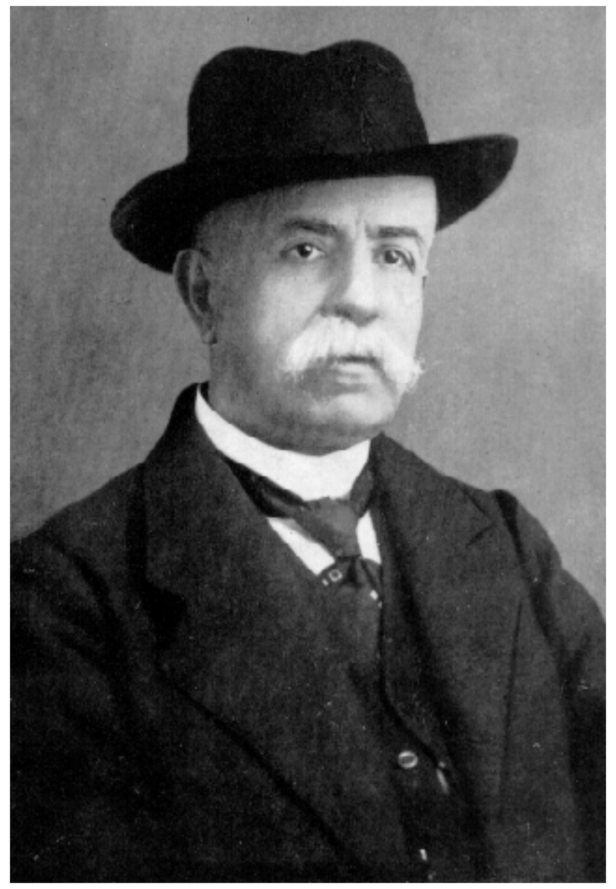

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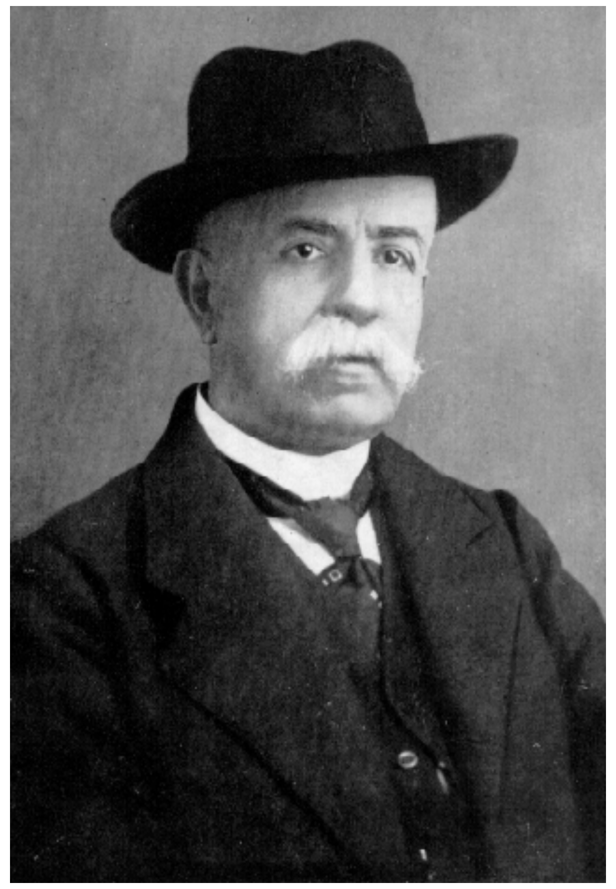

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From 1900 to the present

A relatively general timeline

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- 1922: H. L. Brose's English translation of Weyl's book *Raum, Zeit, Materie* (*Space-Time-Matter*) uses "tensor analysis."

So what is a “tensor” anyway?

Tensors are many different things to many different people

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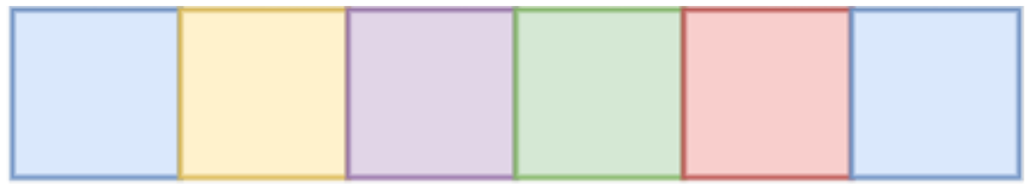

$\mathbf{x} \in \mathrm{R}^m$

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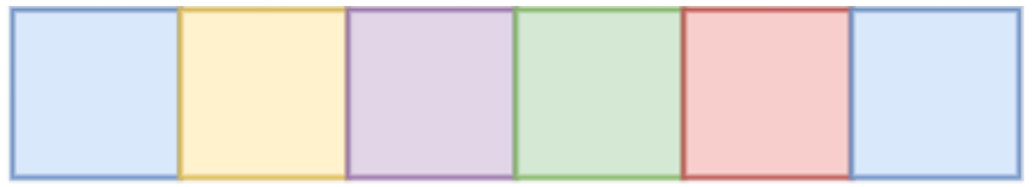

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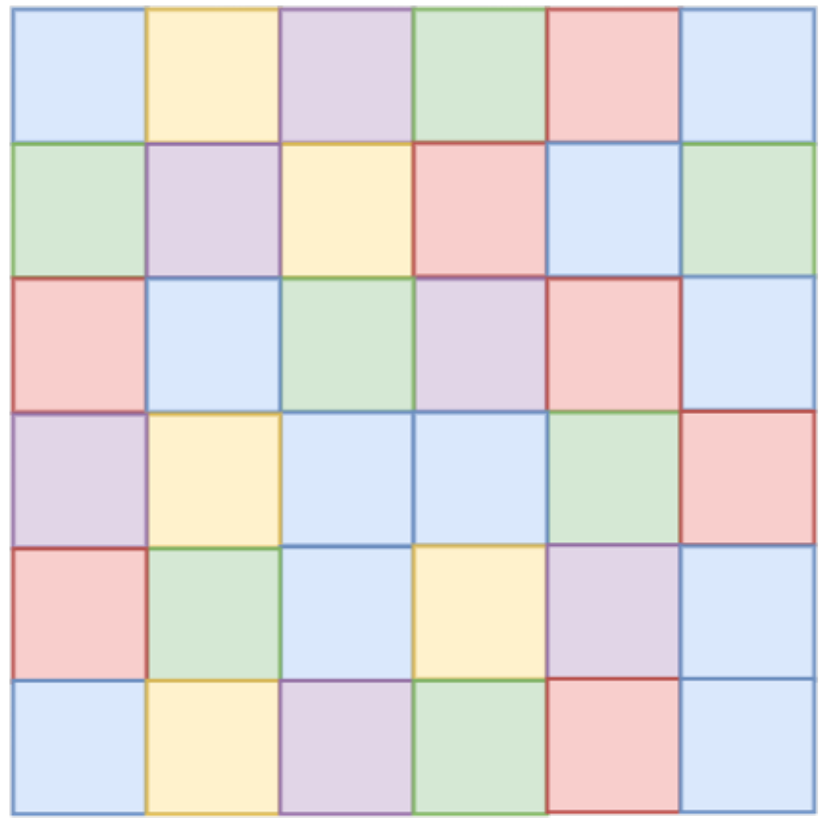

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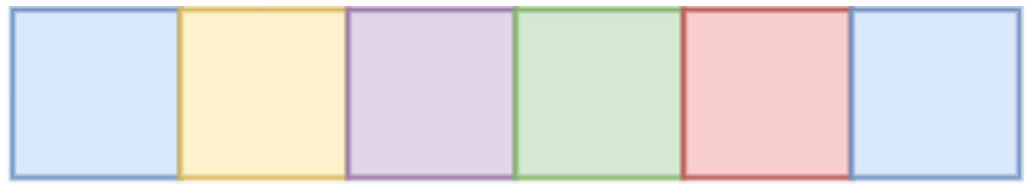

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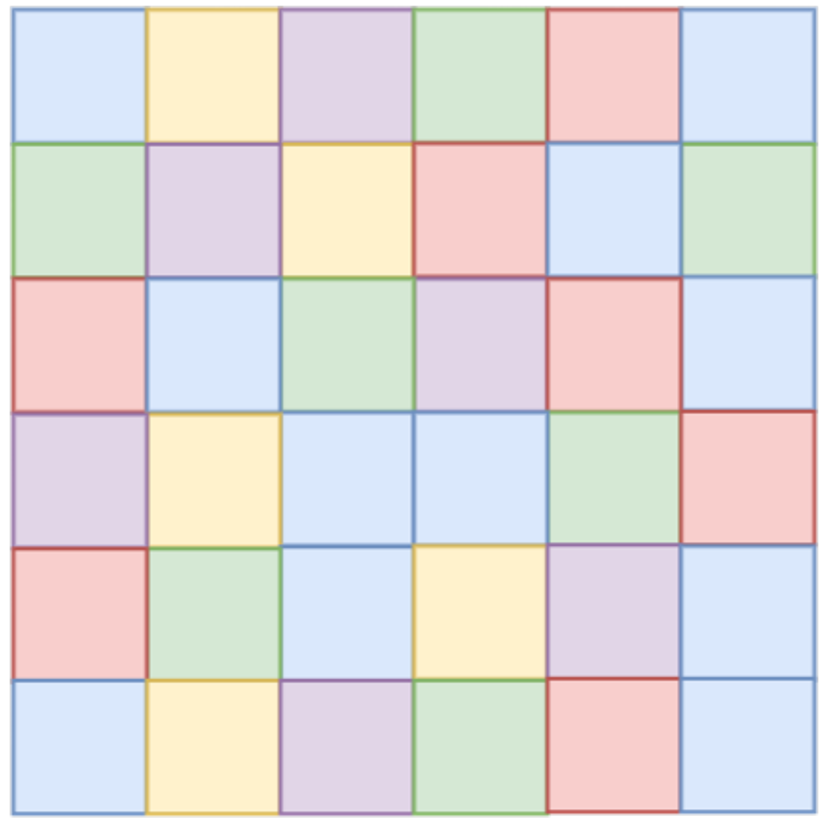

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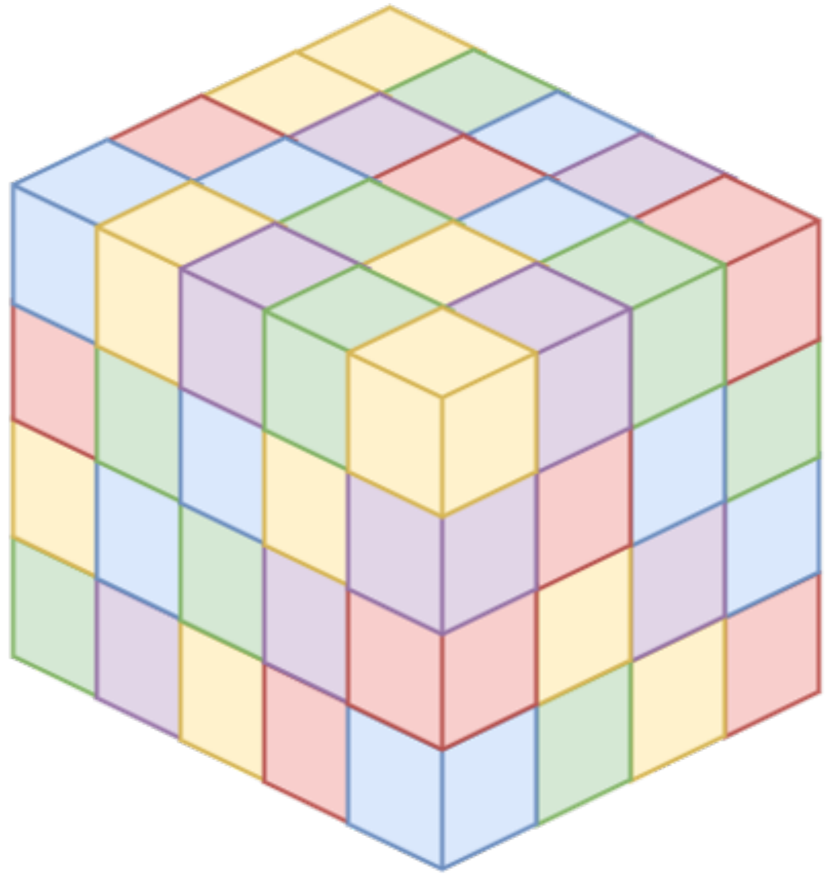

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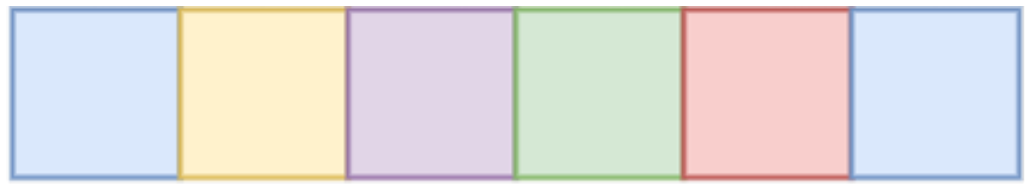

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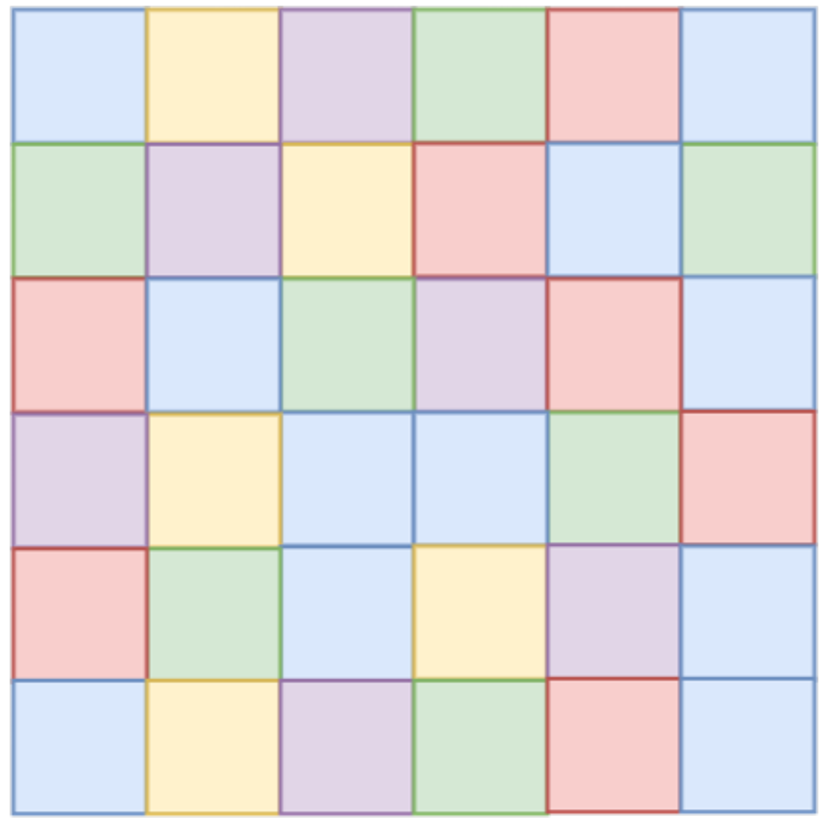

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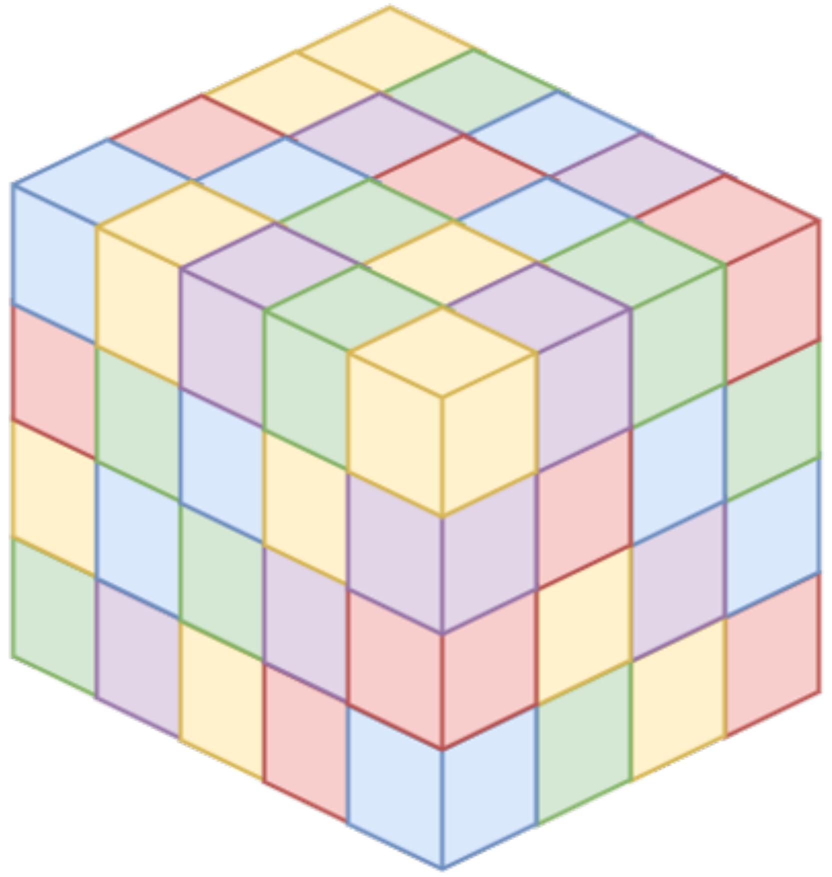

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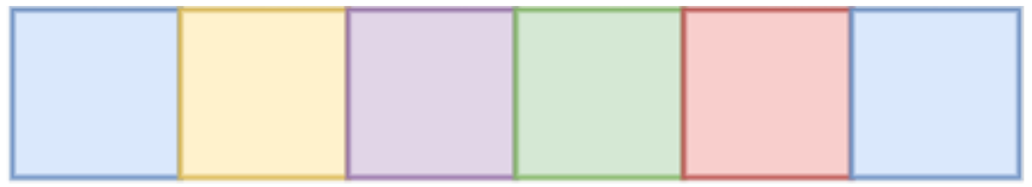

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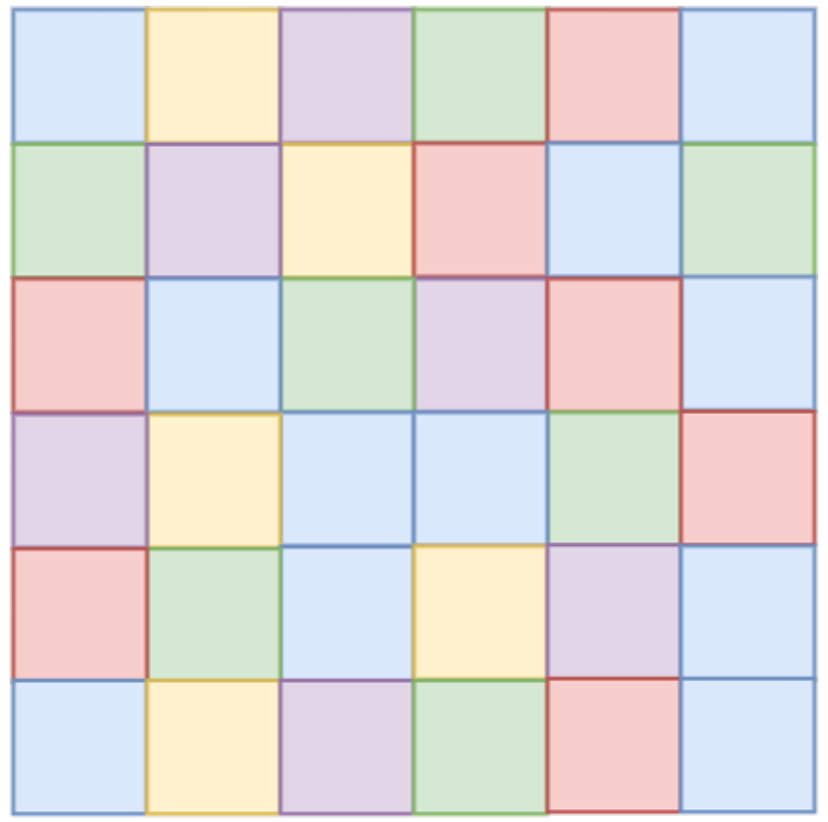

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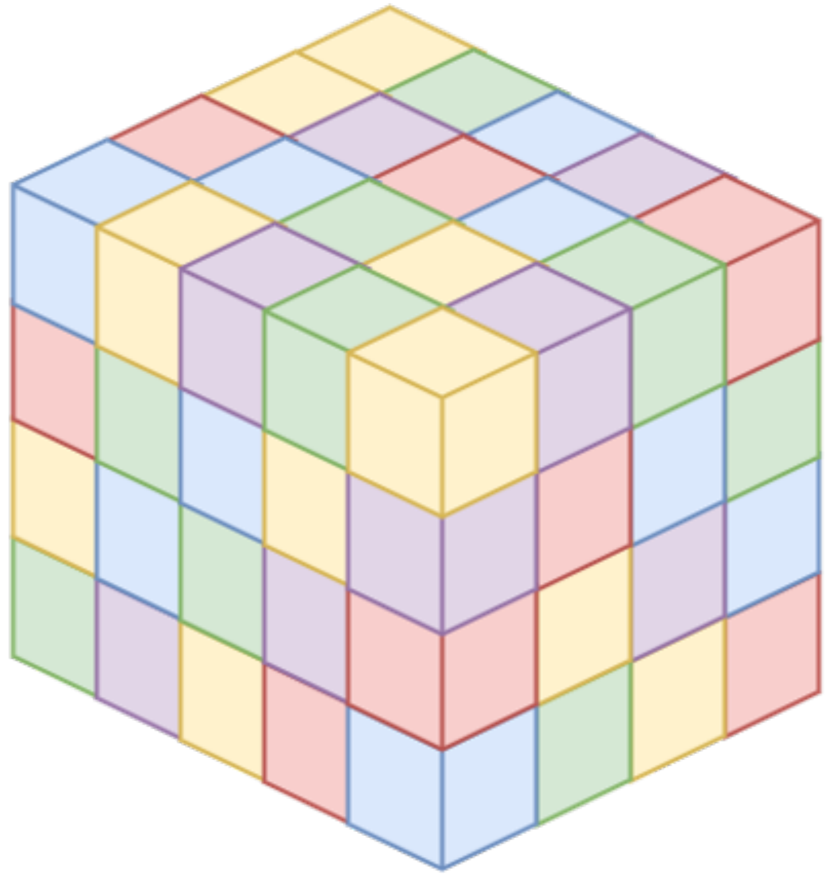

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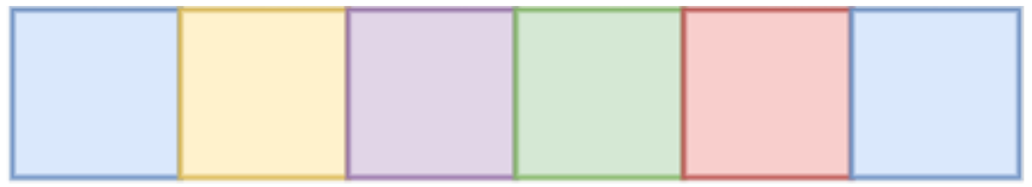

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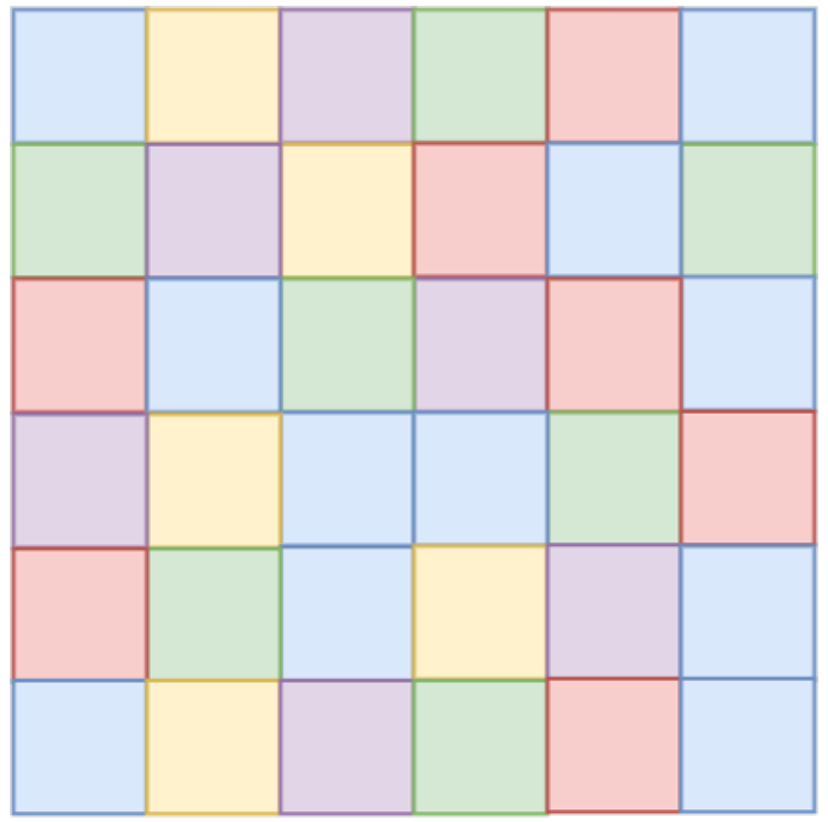

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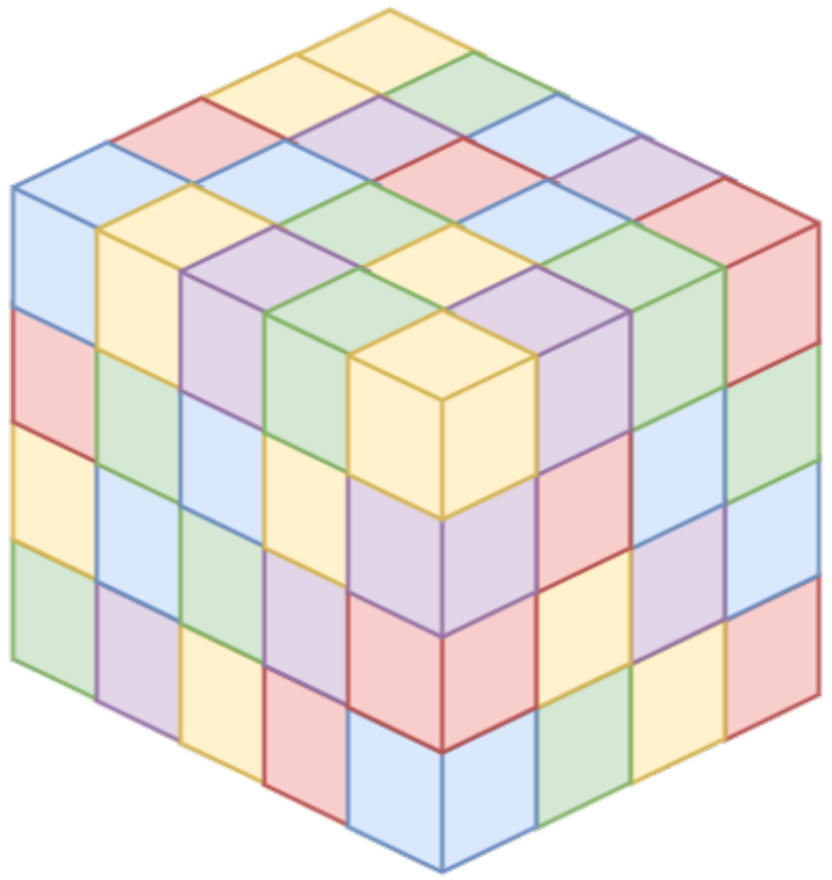

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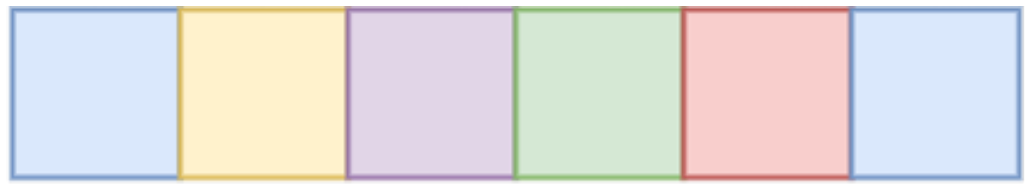

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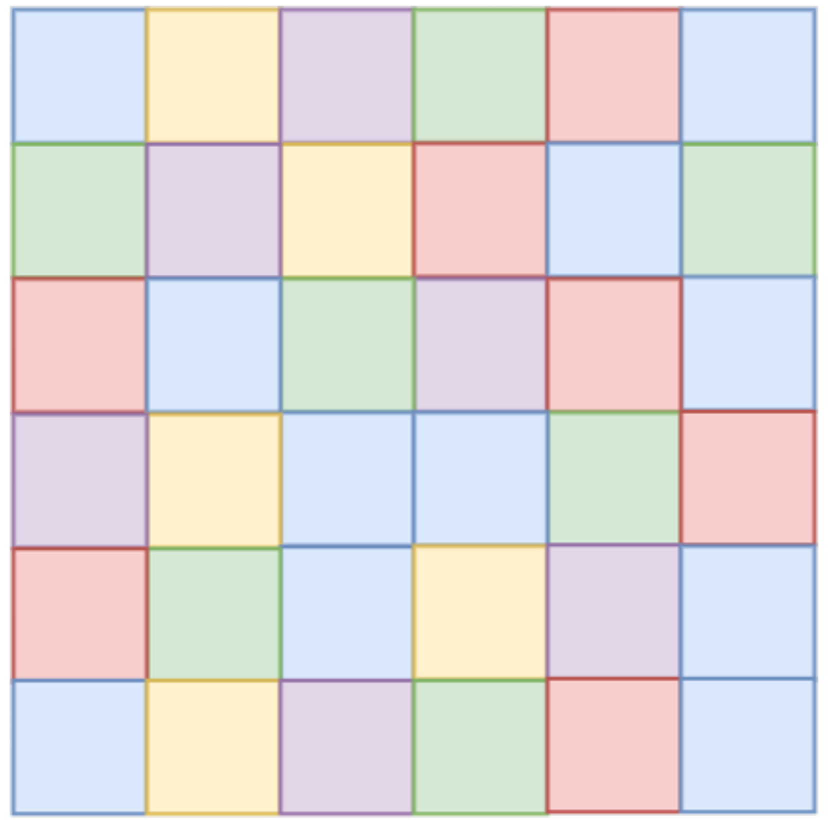

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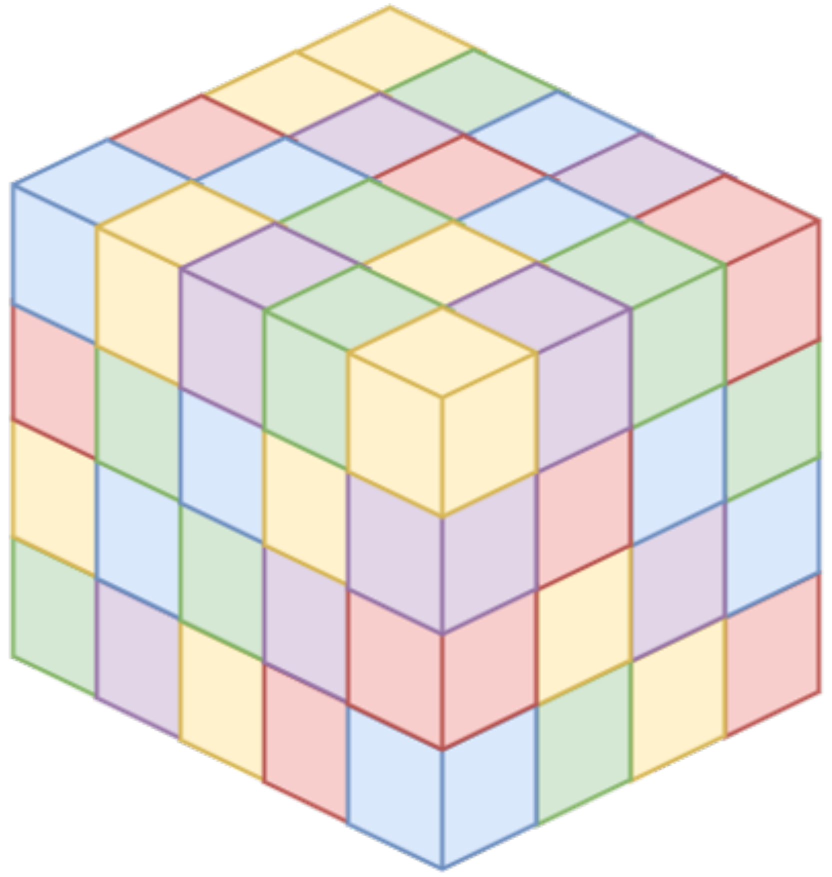

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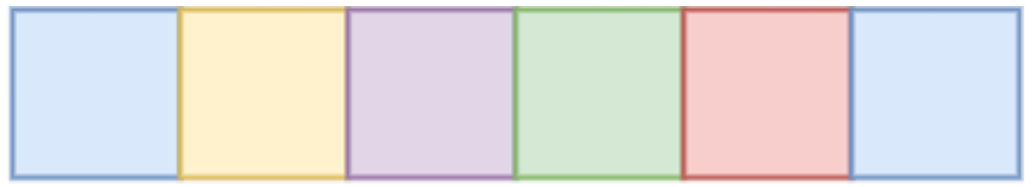

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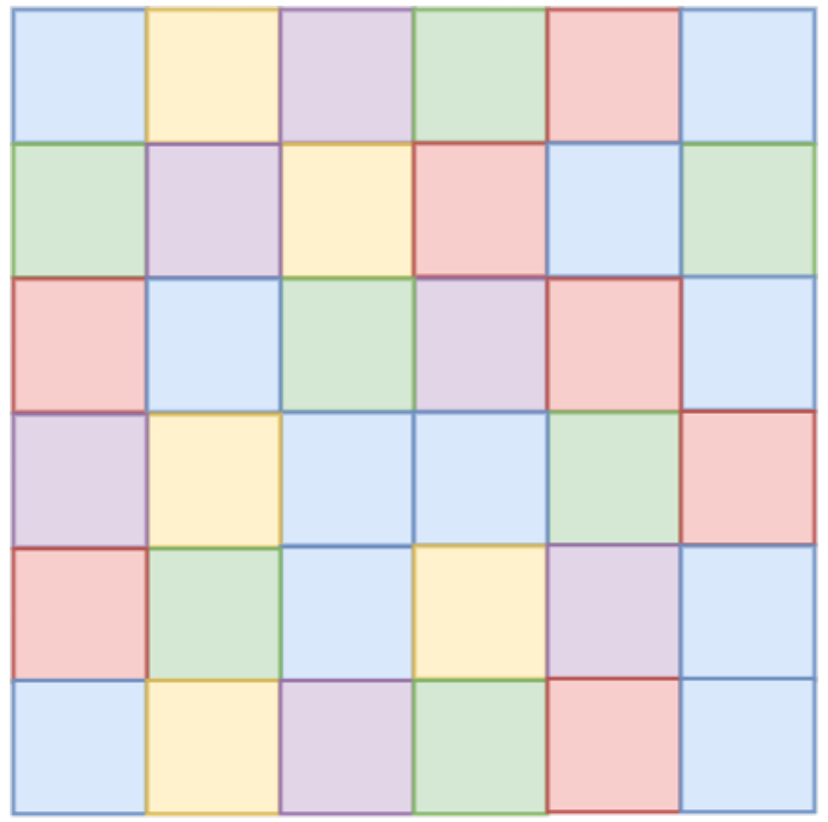

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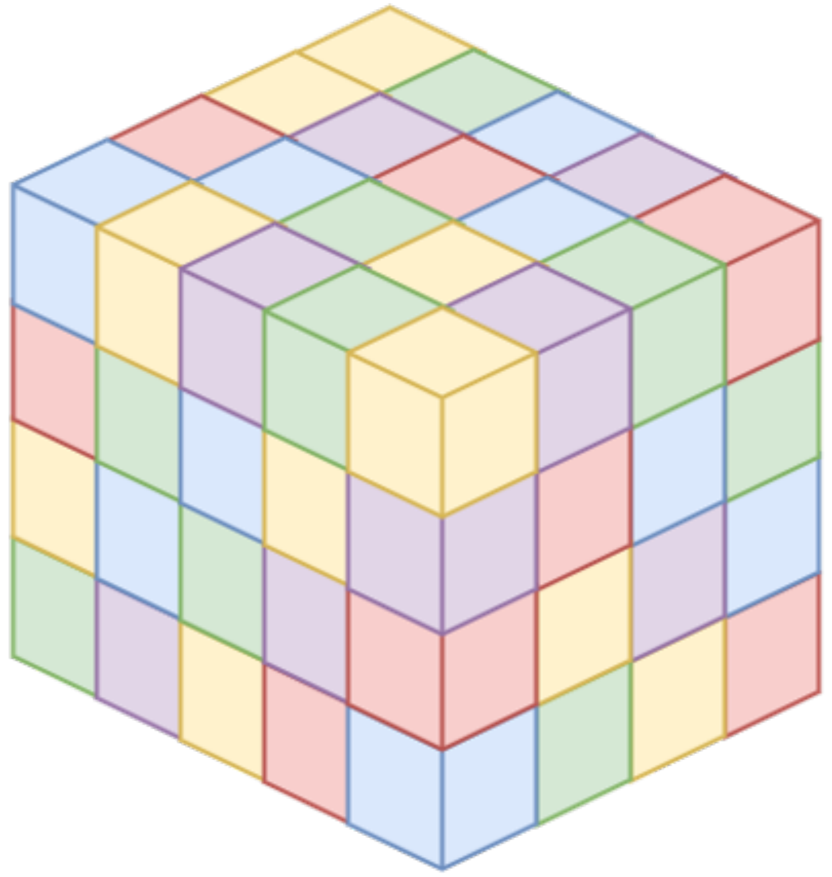

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Where do we see tensor-valued data?

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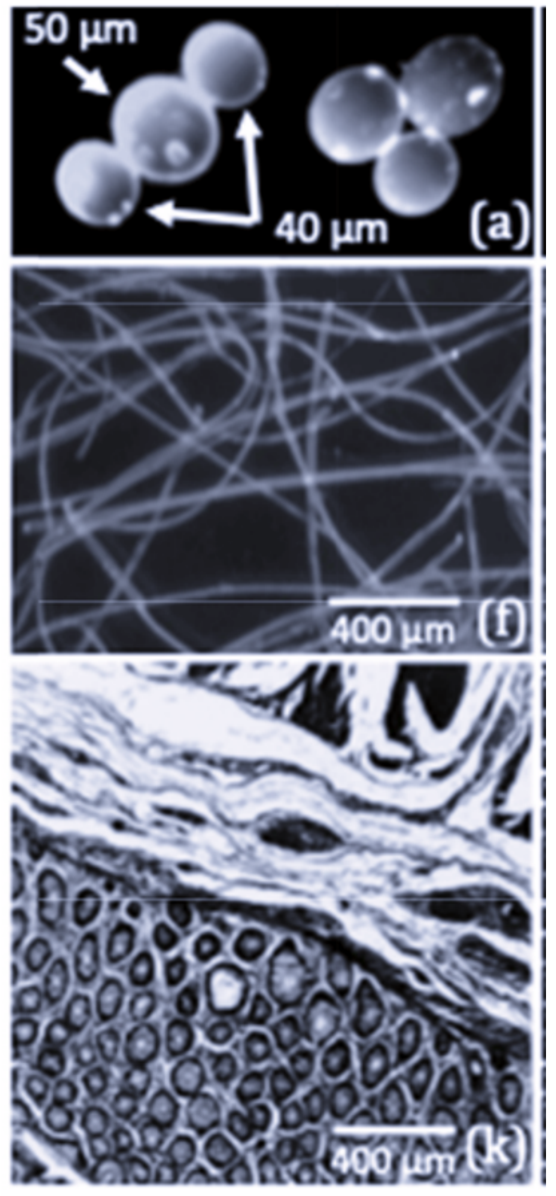

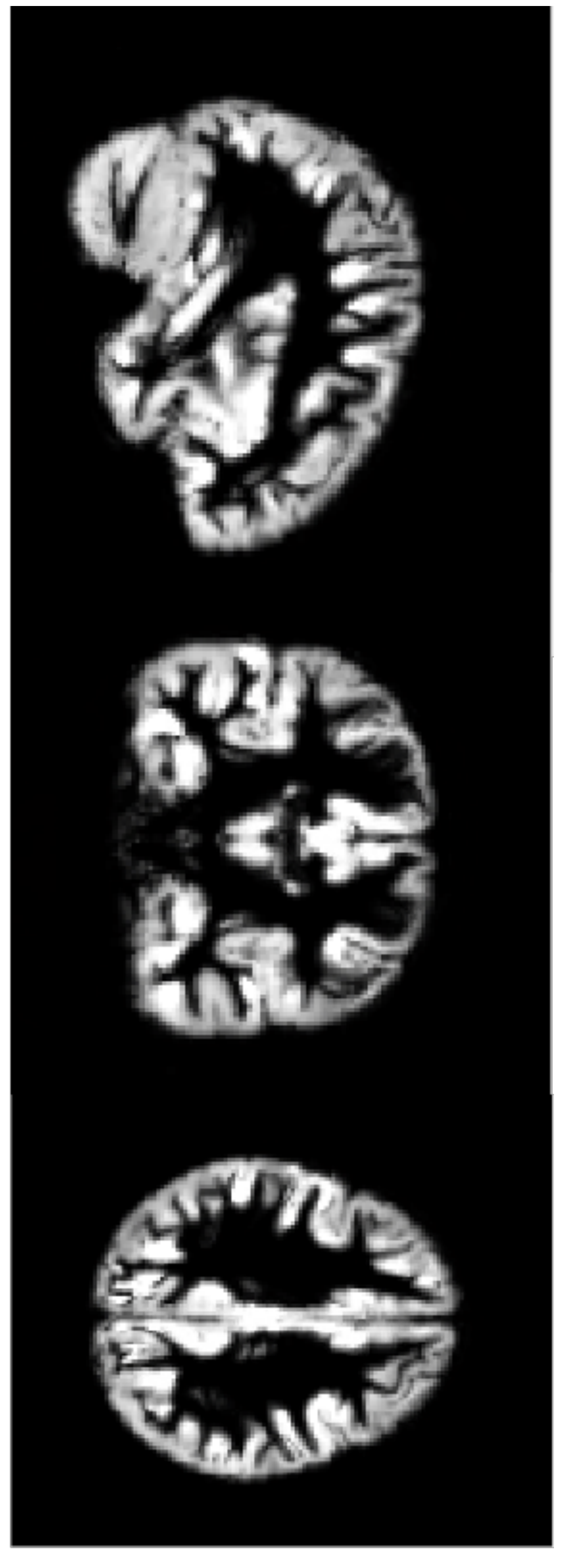

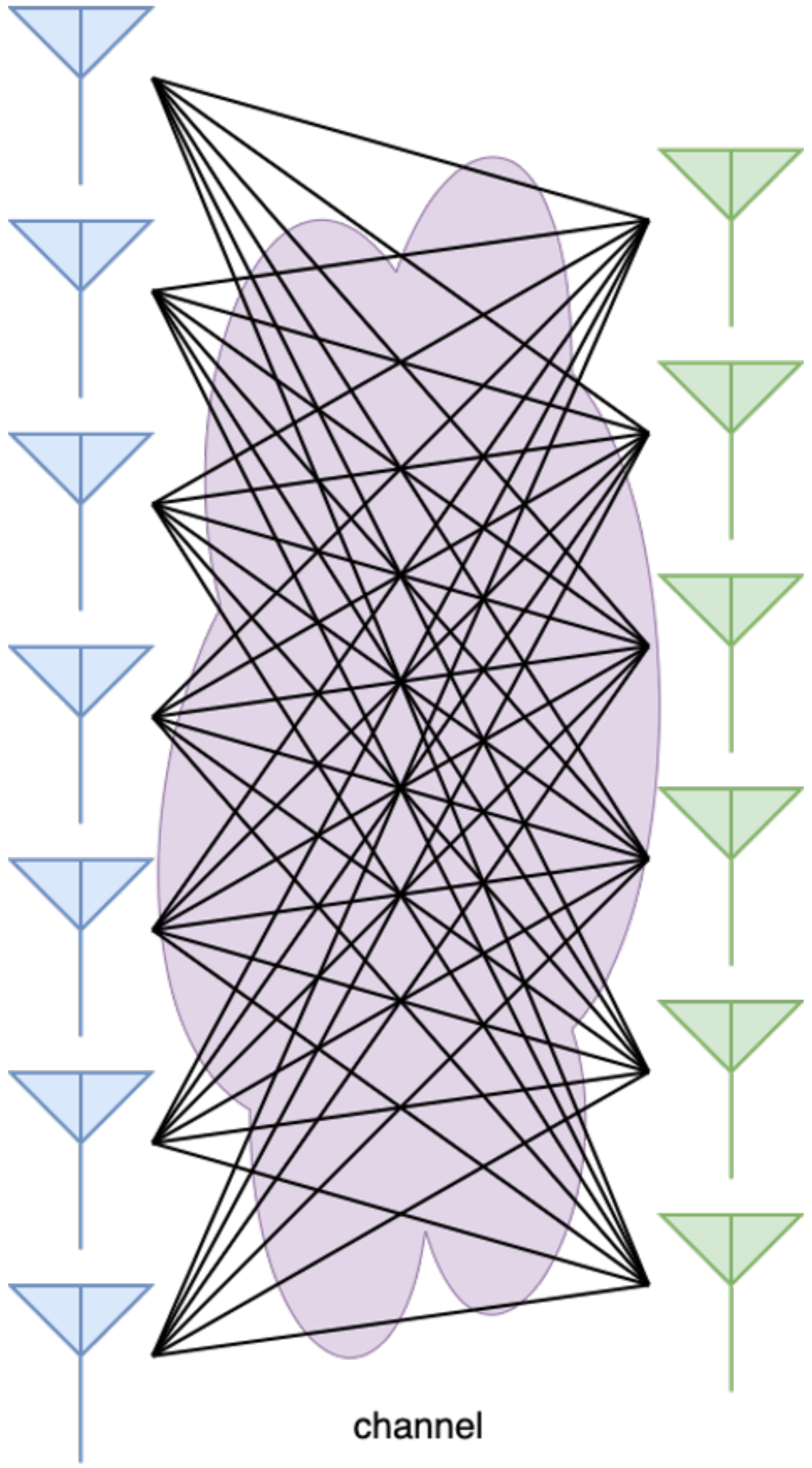

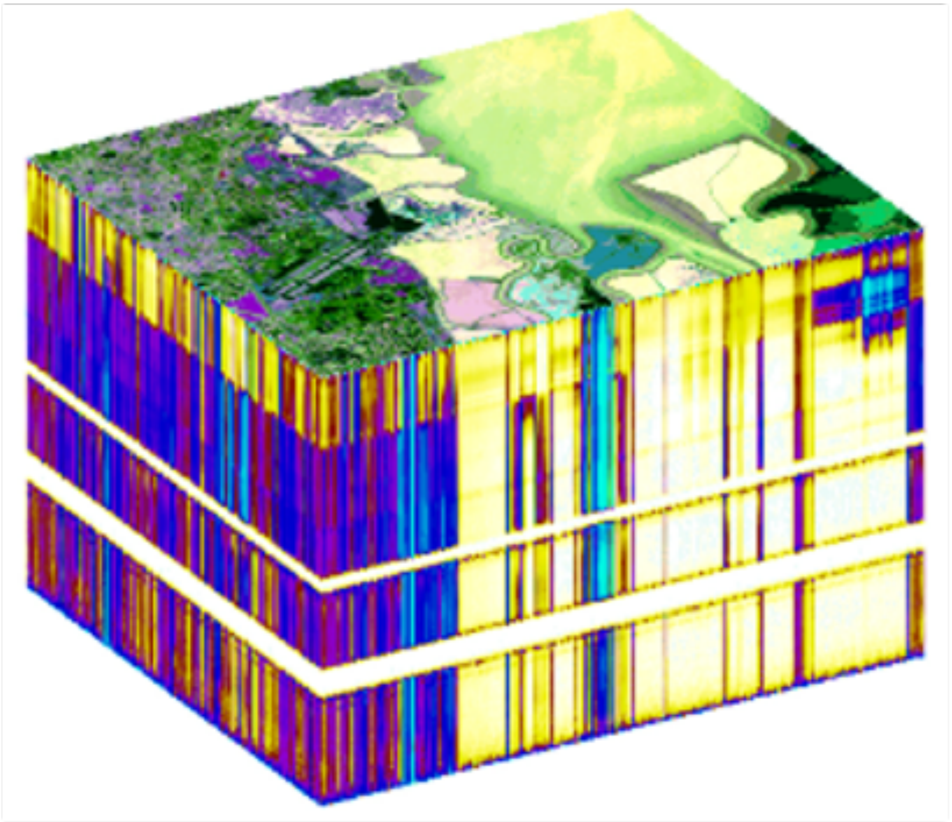

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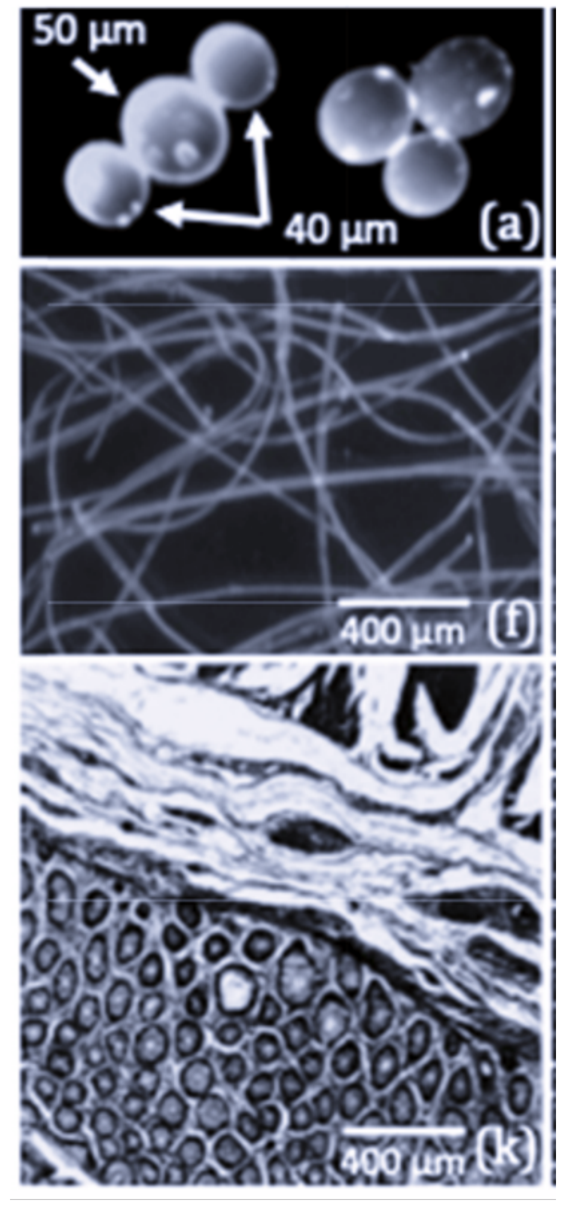

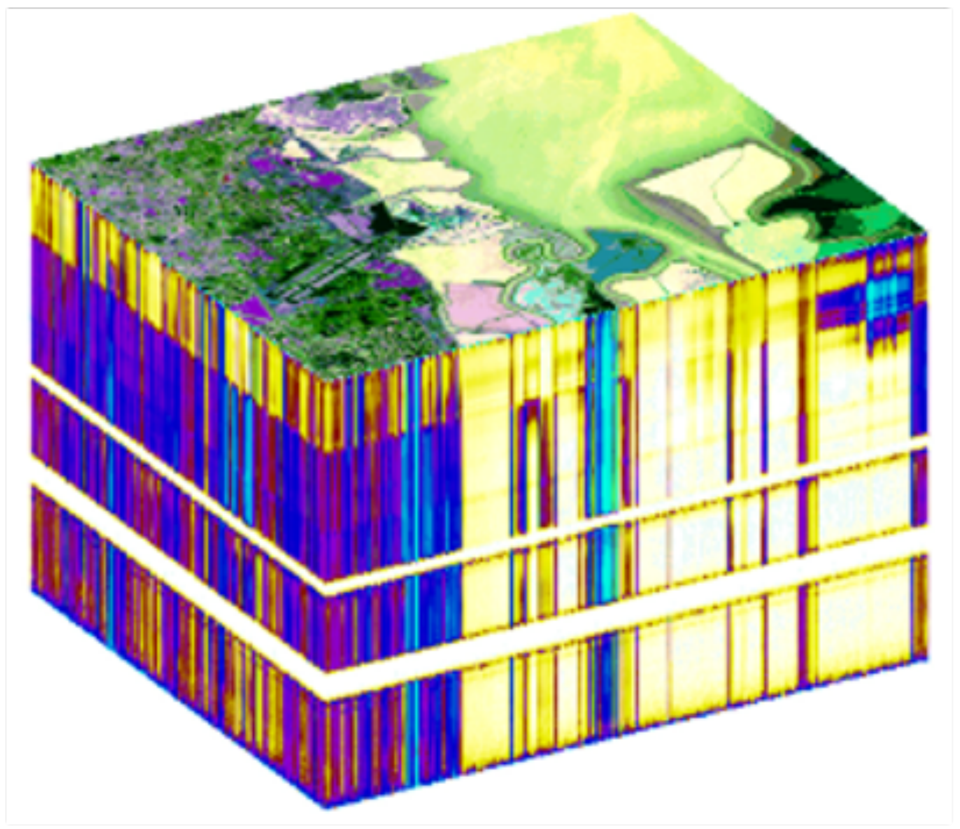

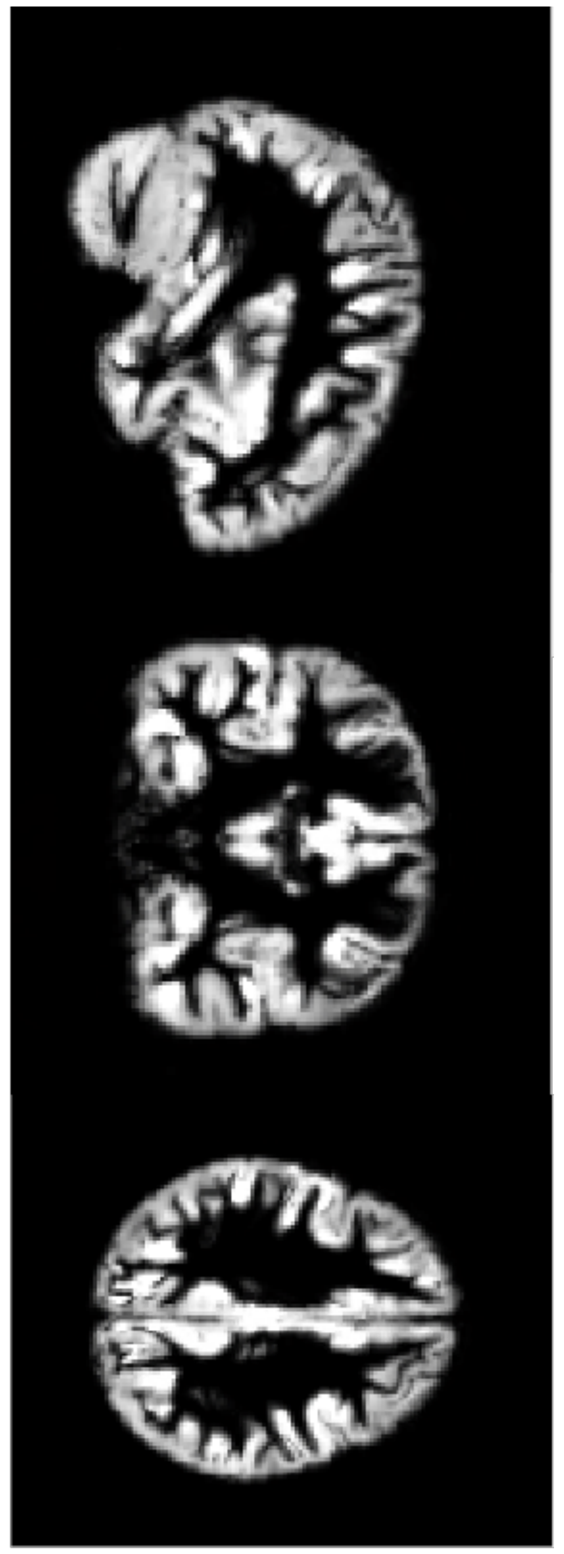

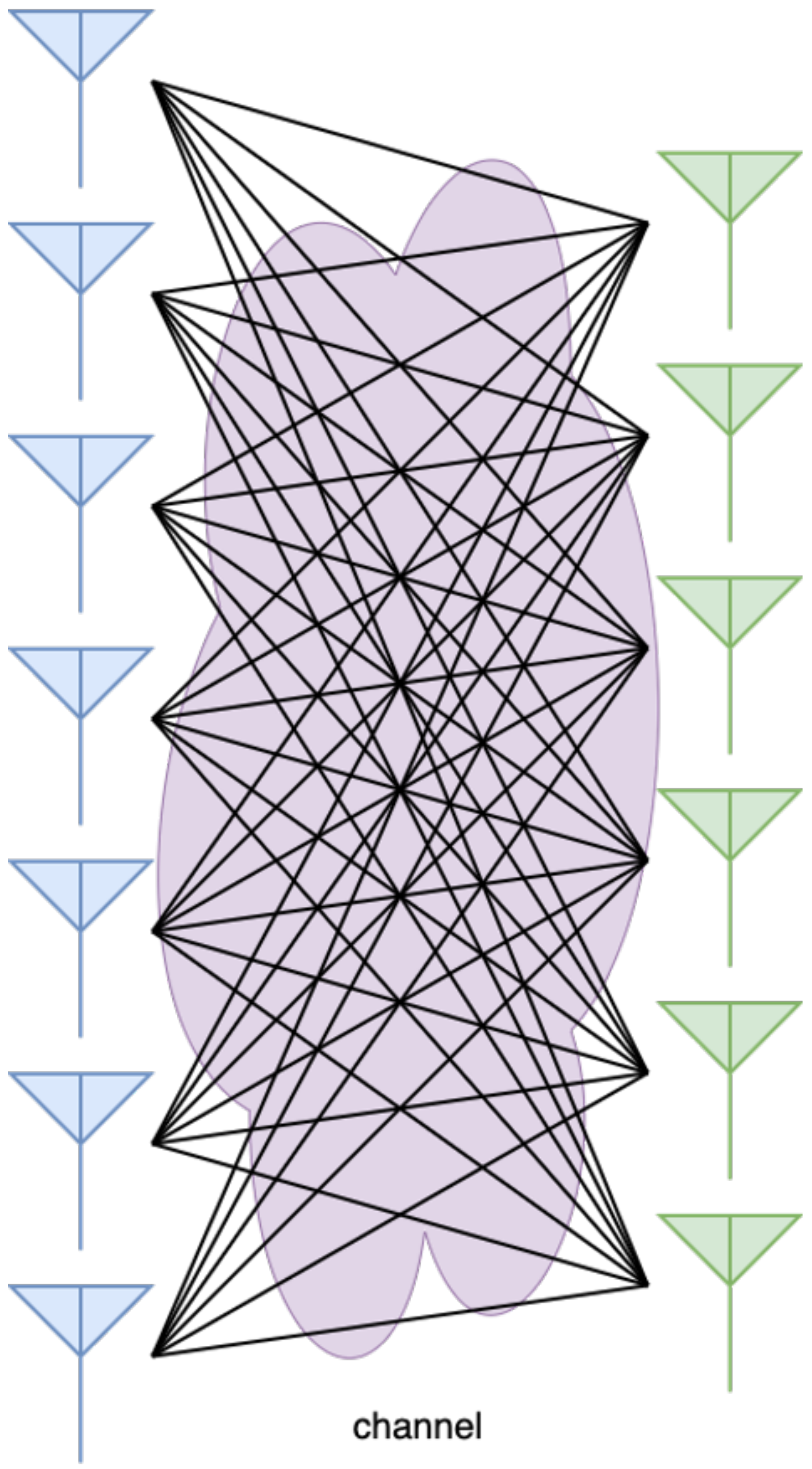

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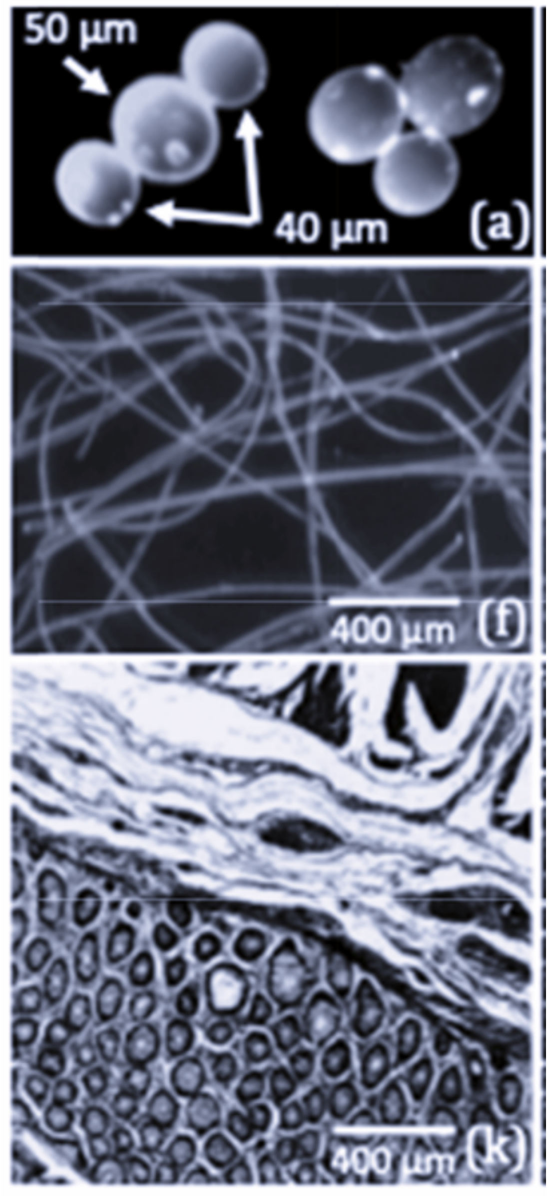

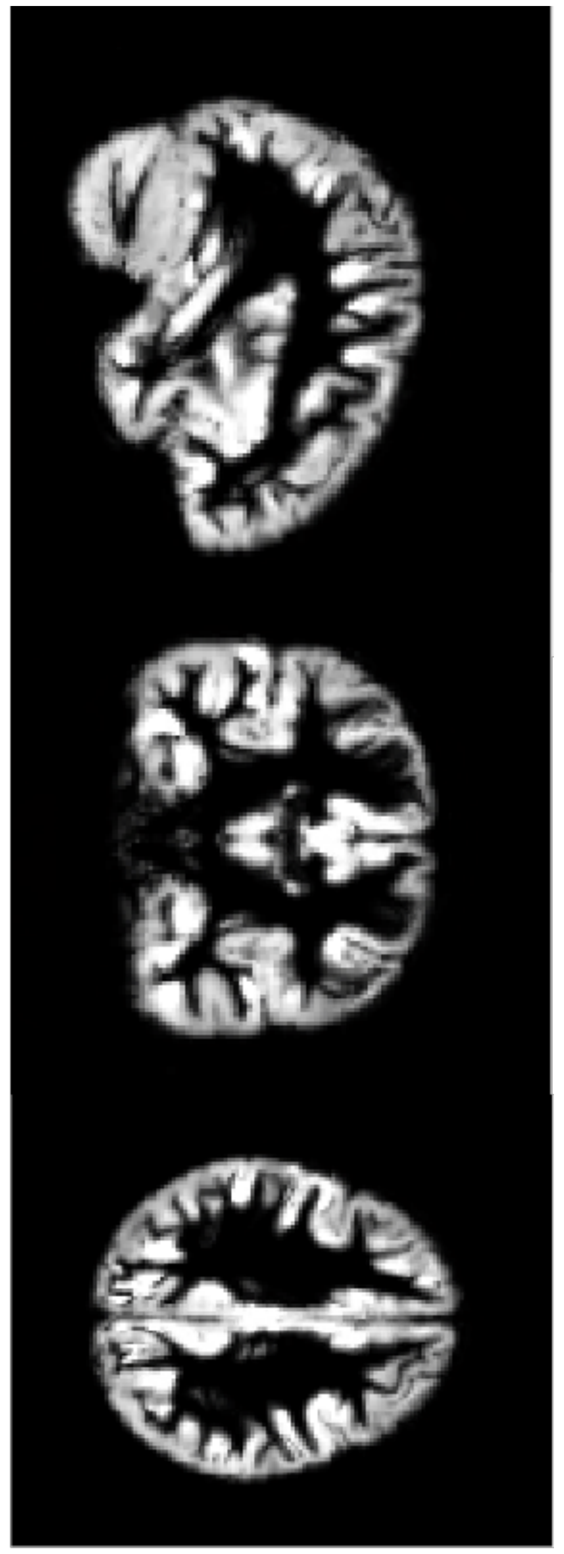

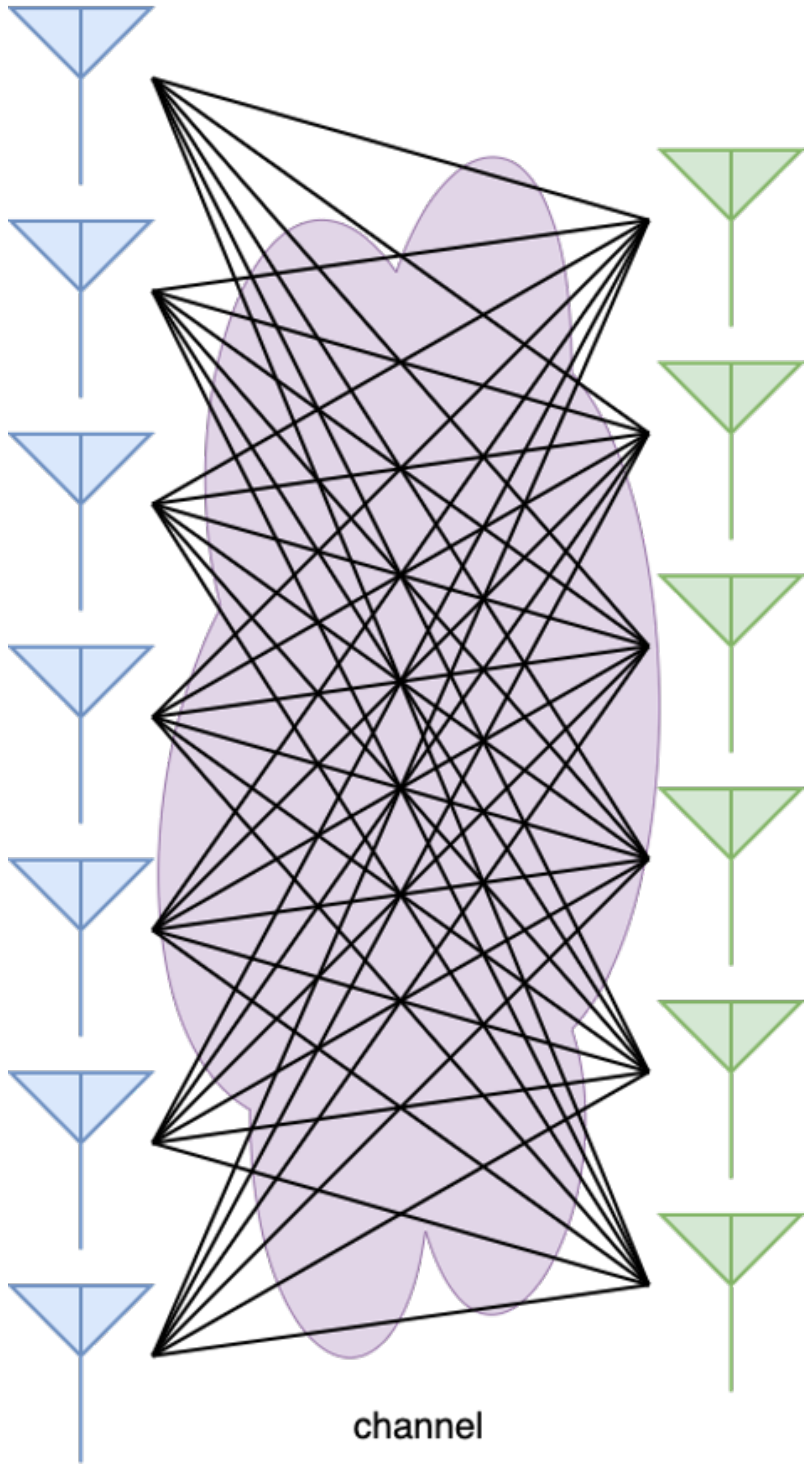

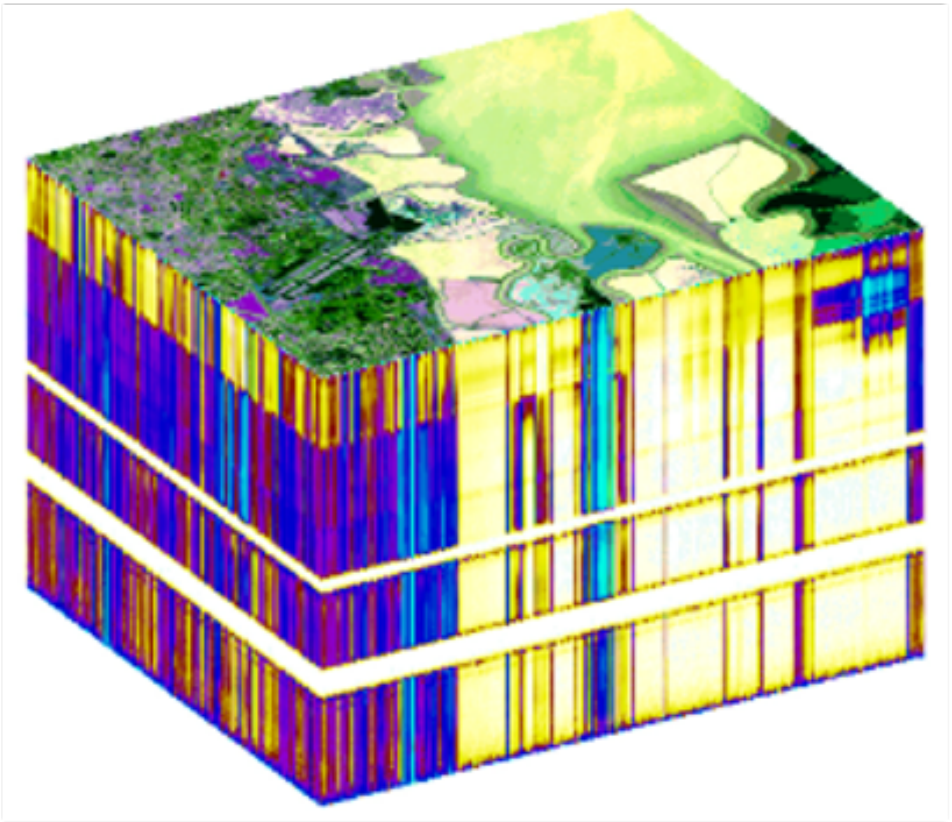

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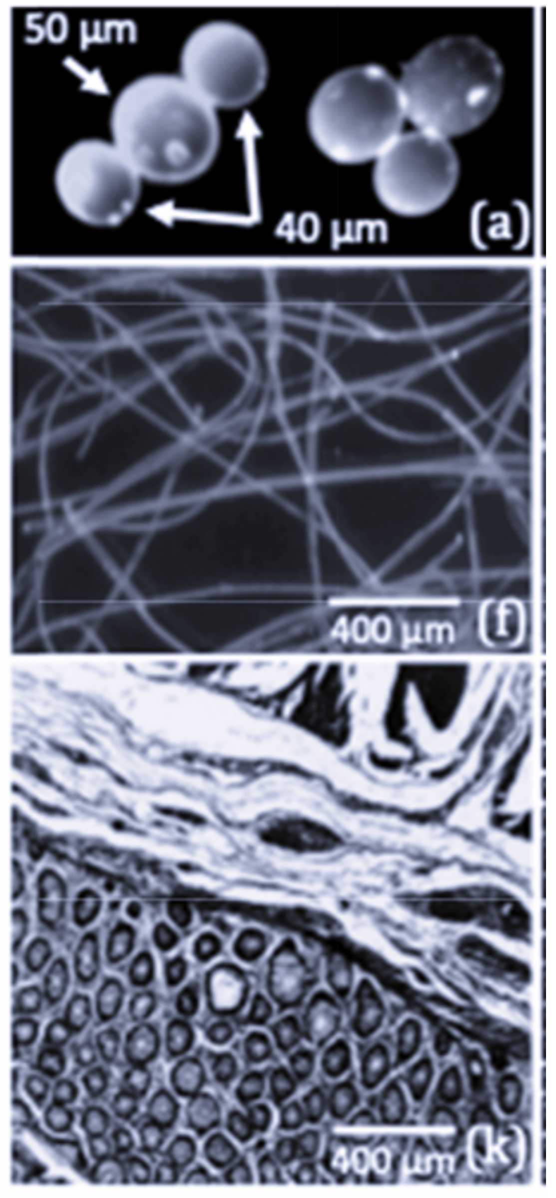

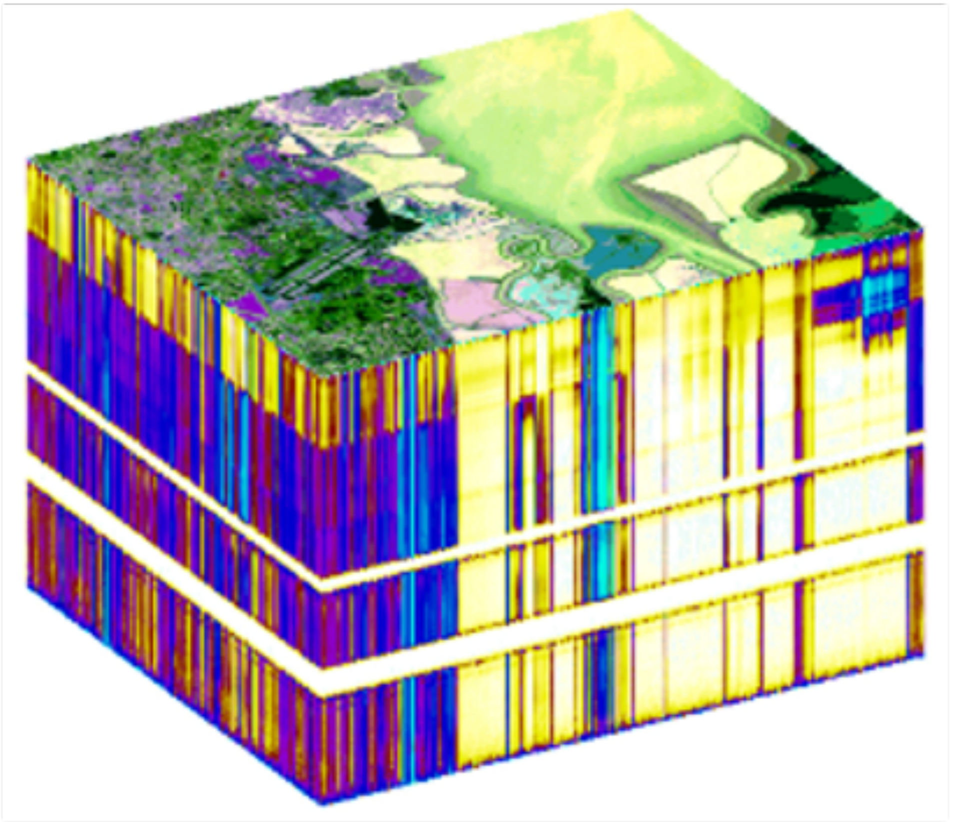

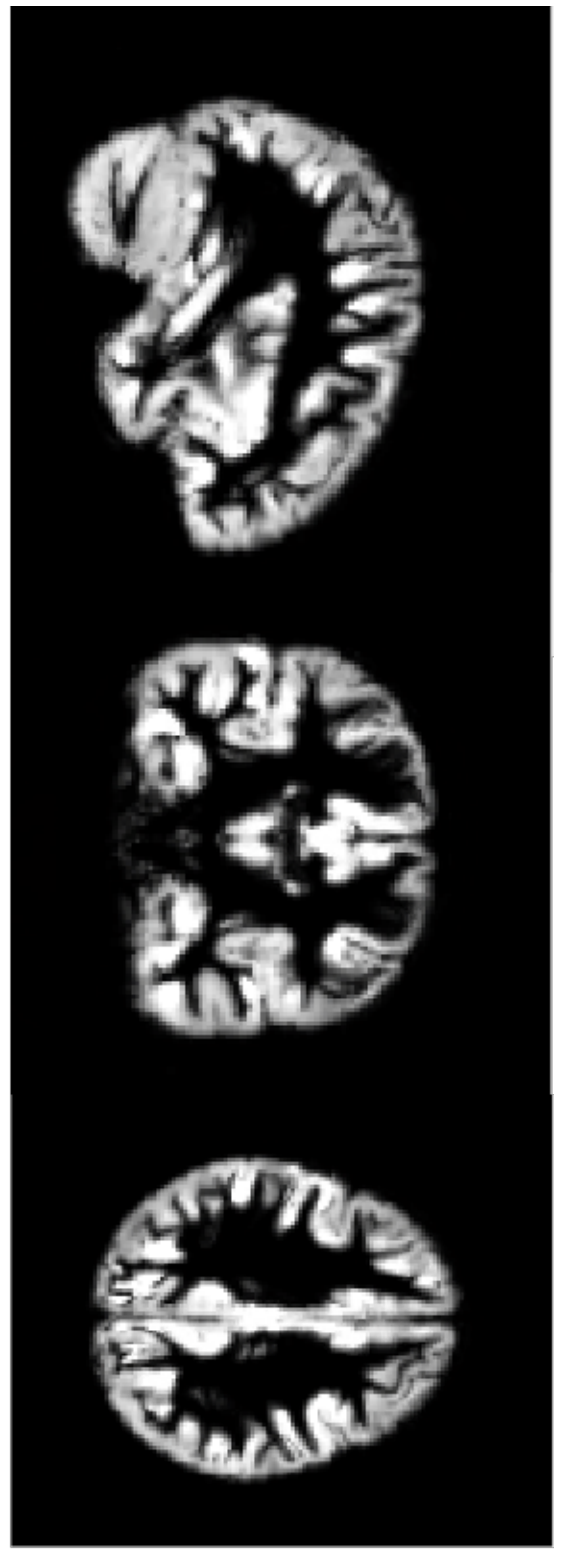

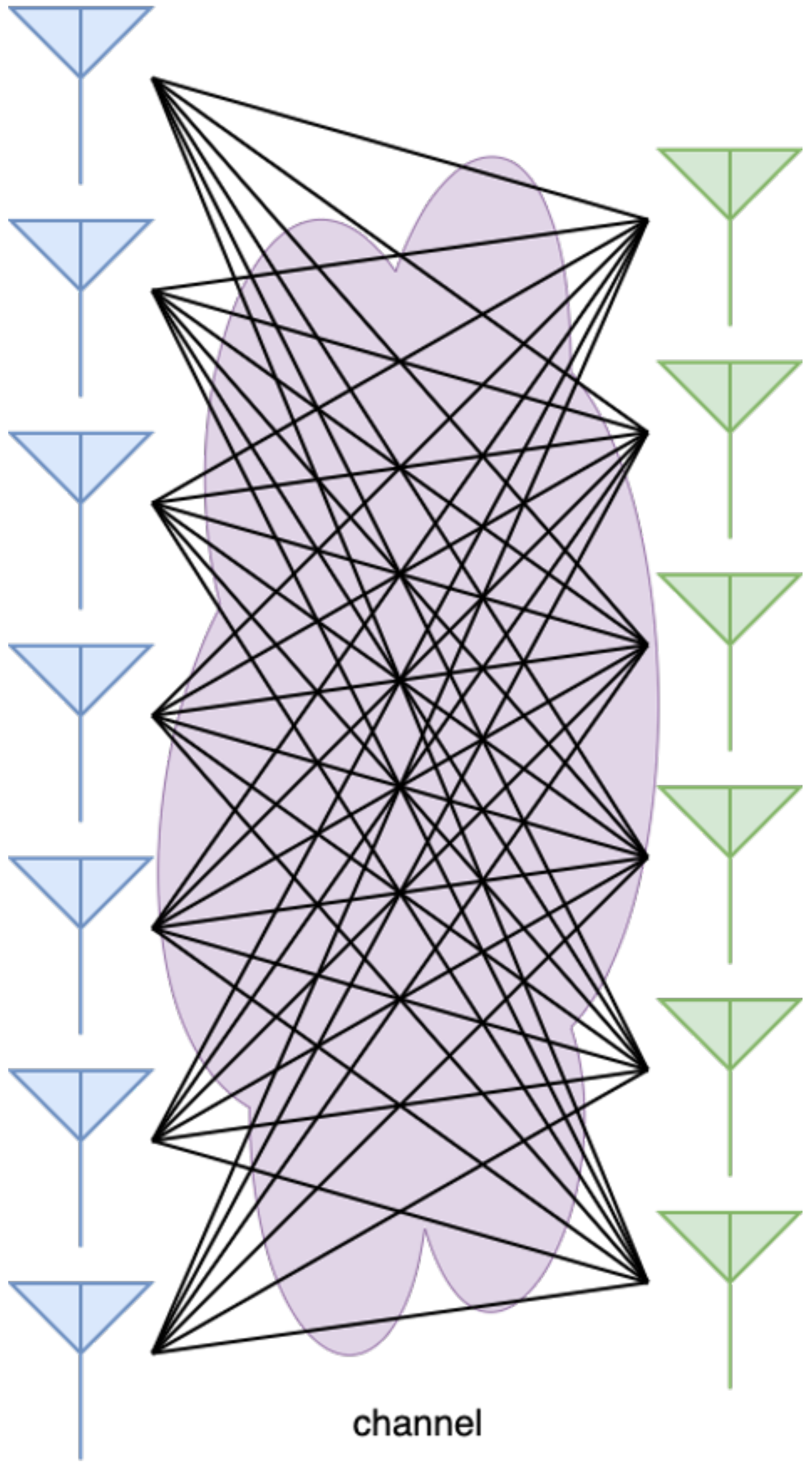

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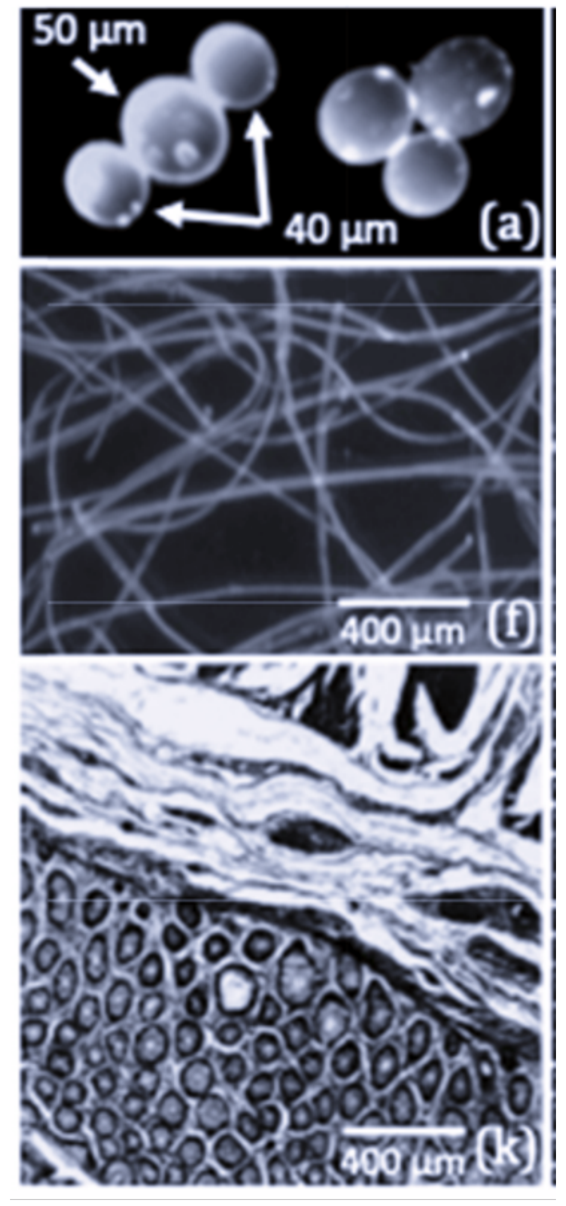

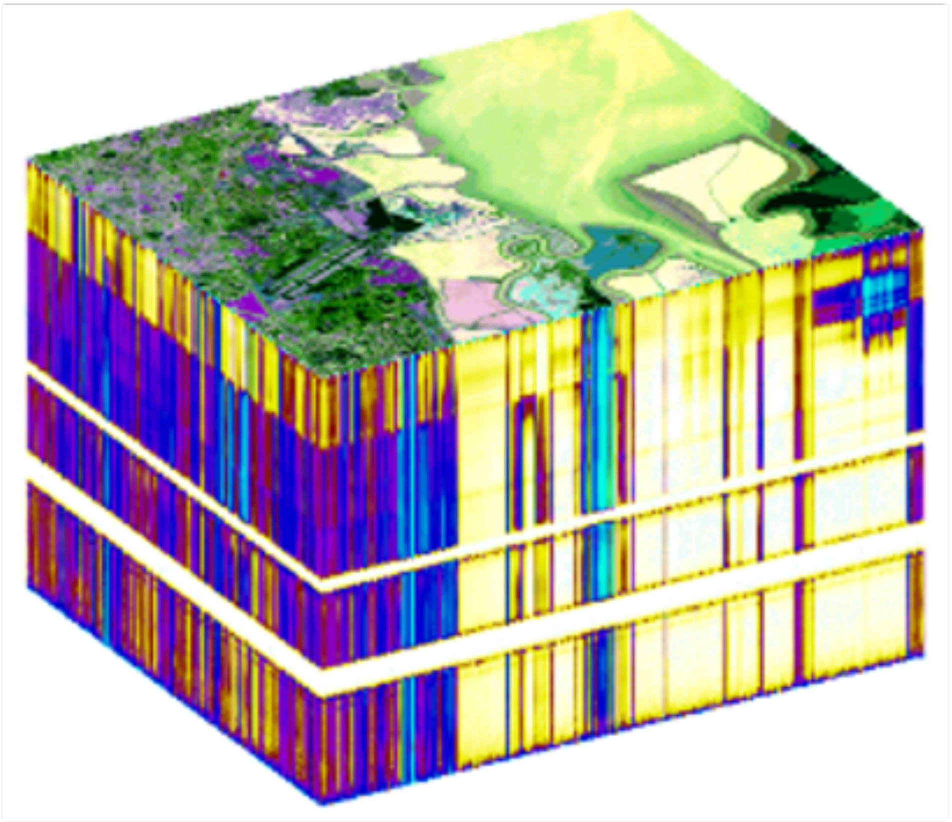

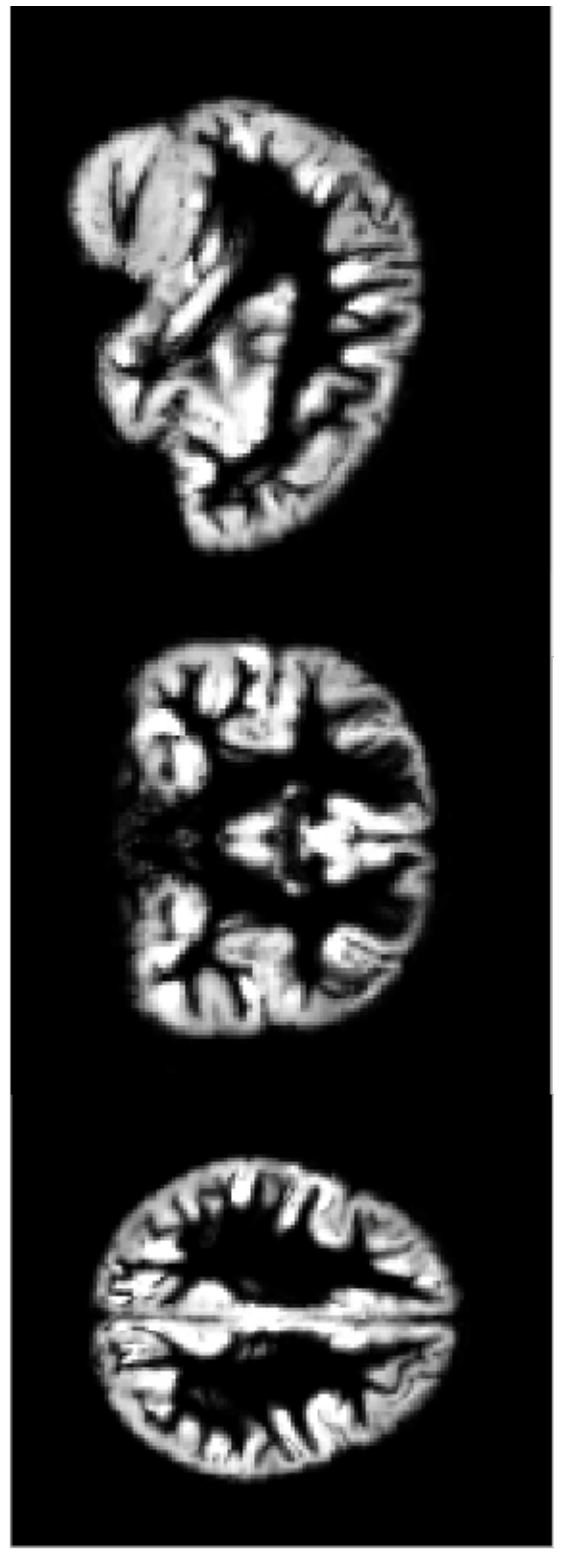

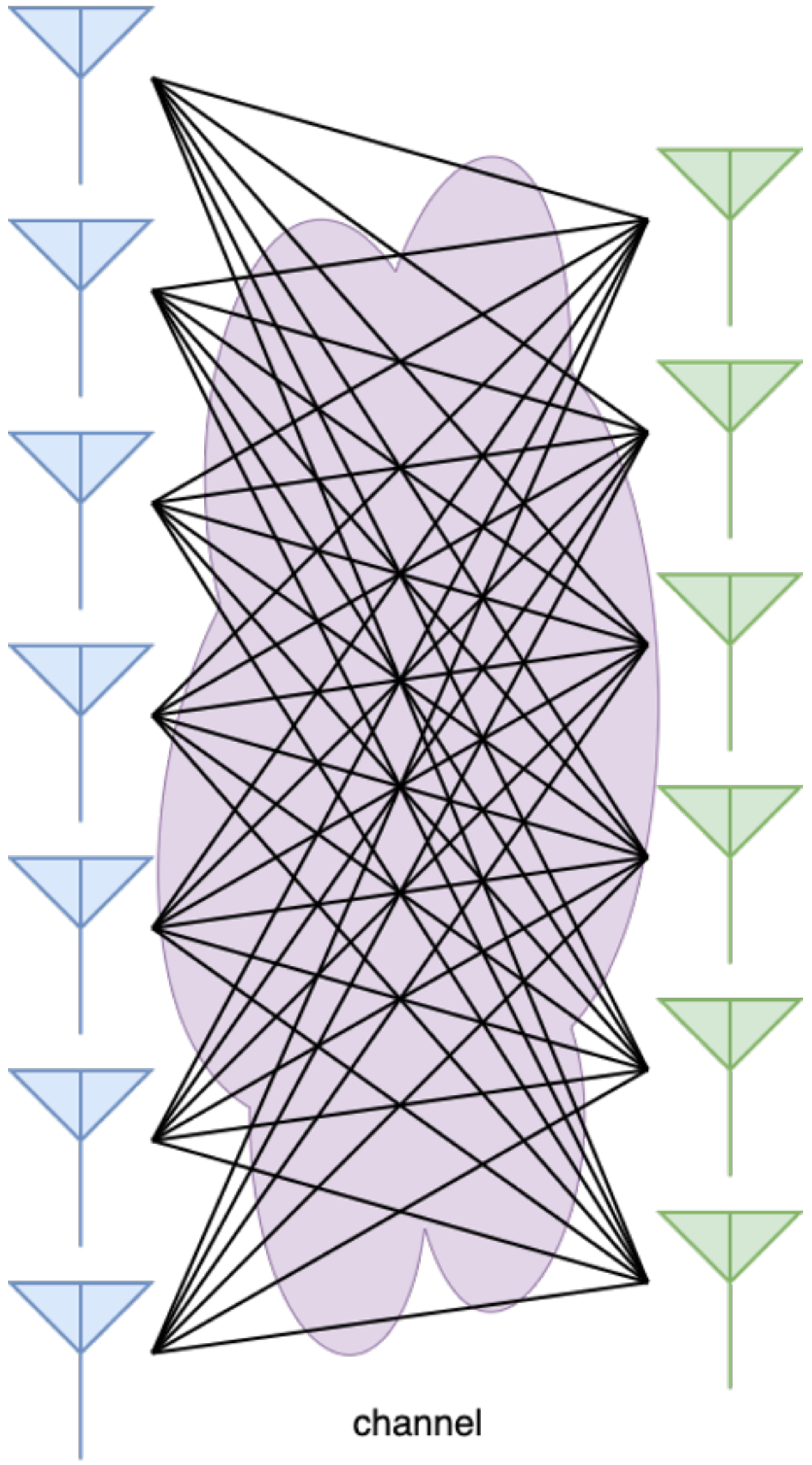

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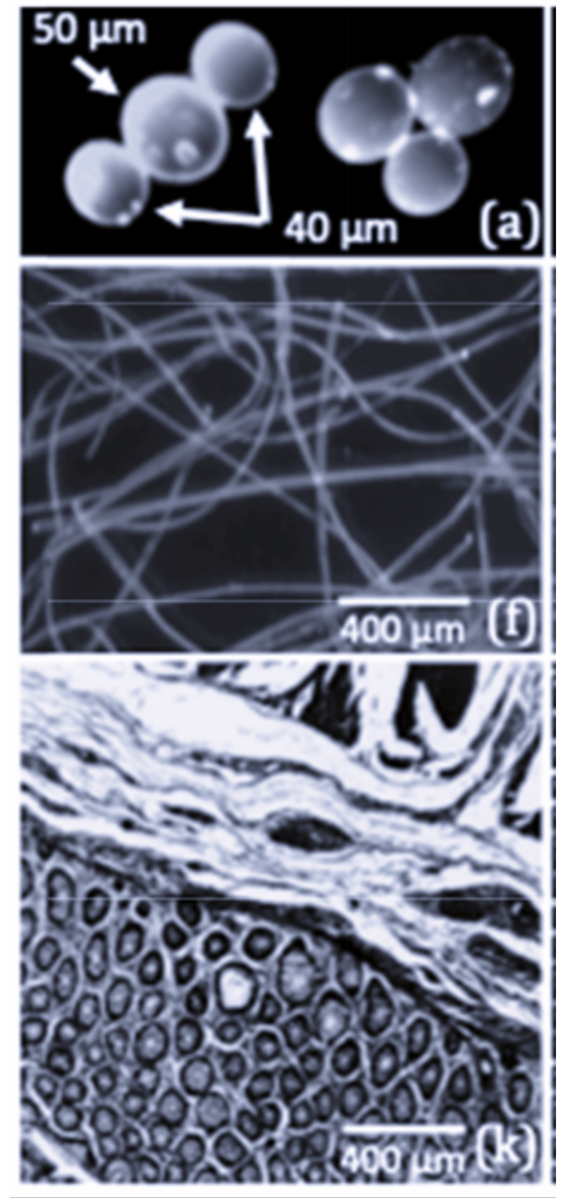

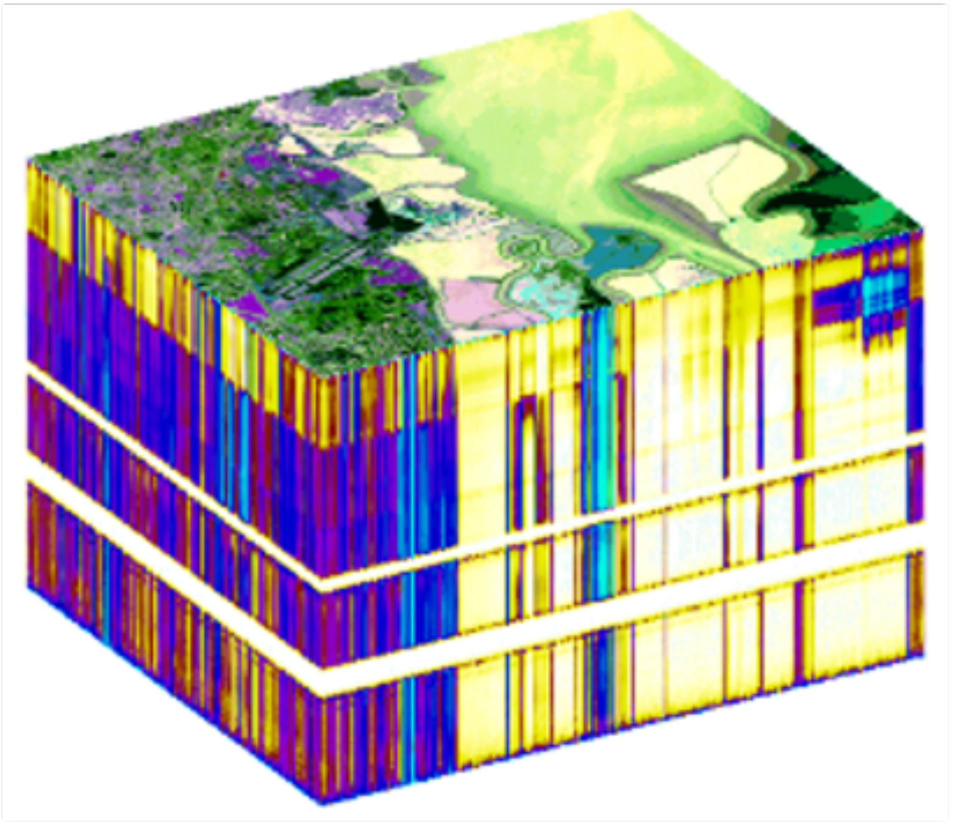

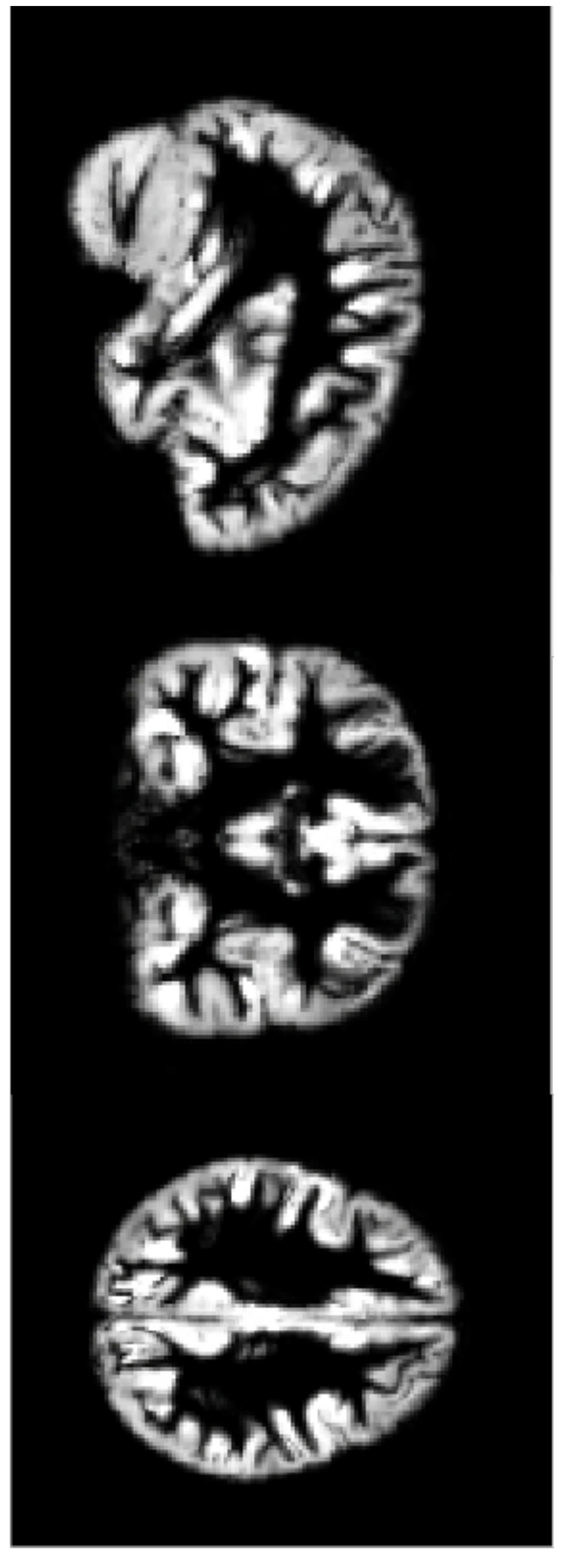

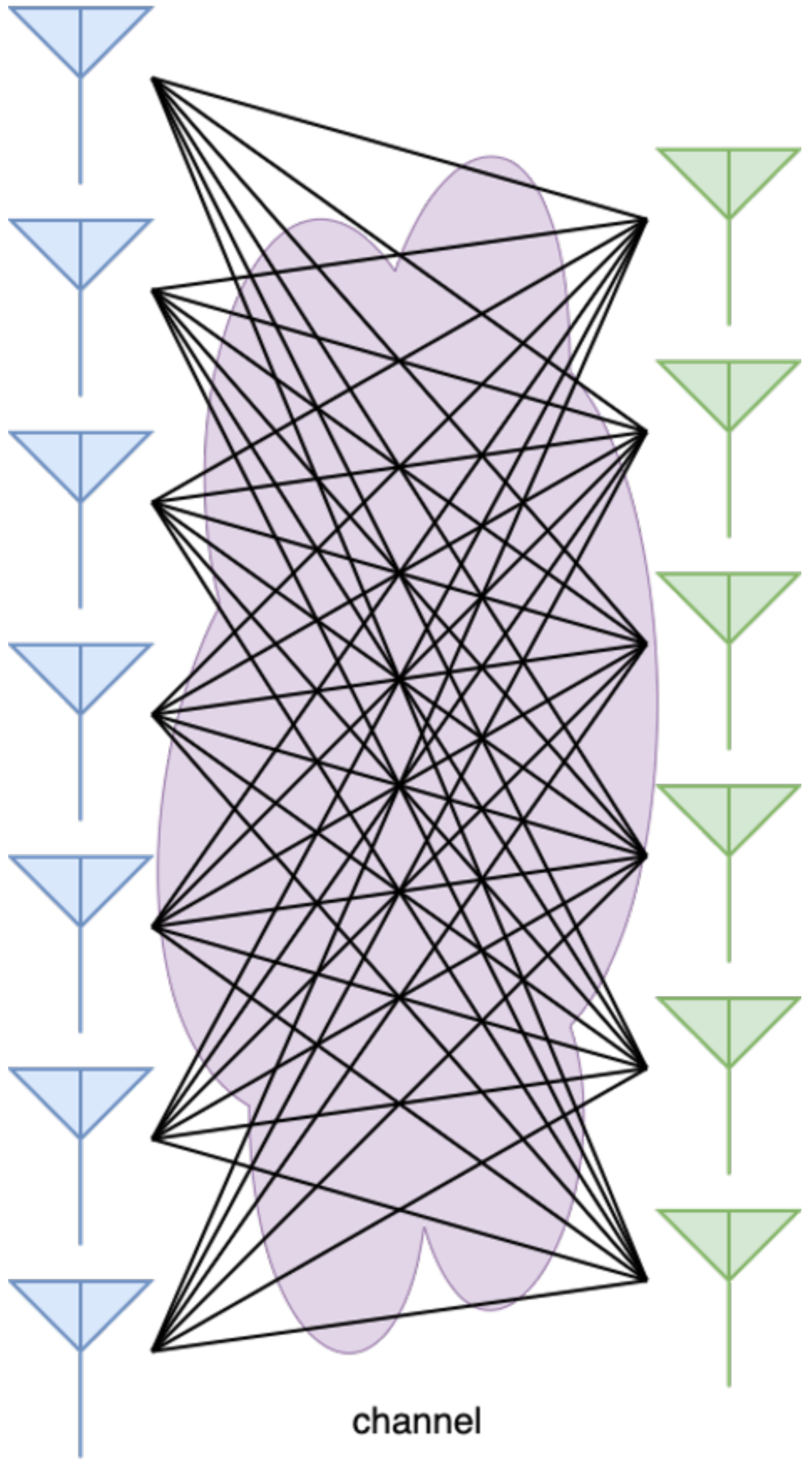

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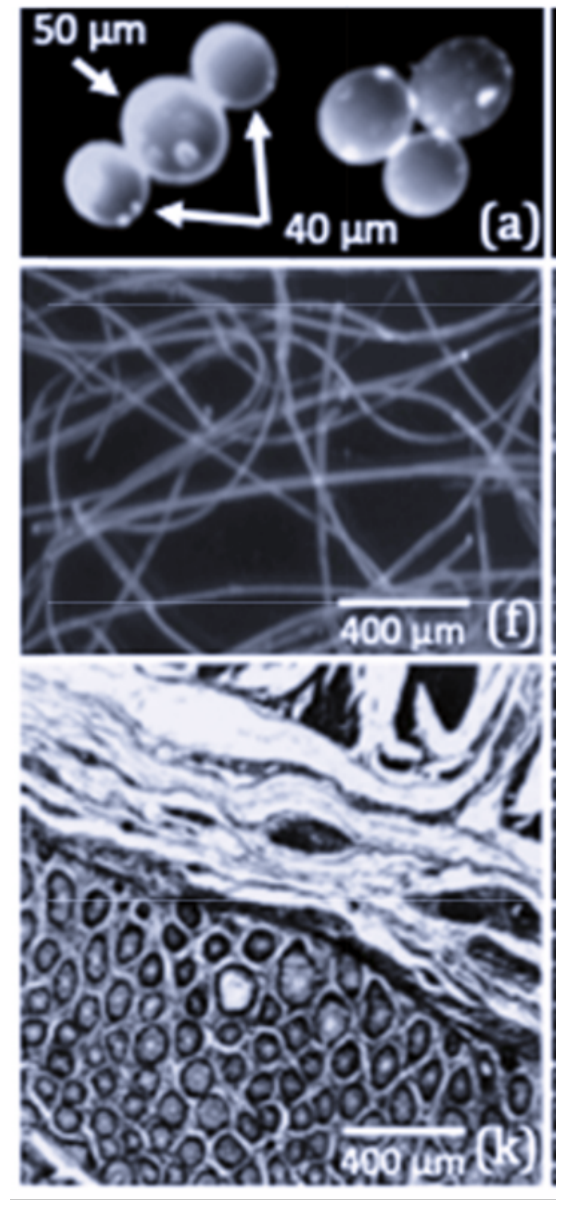
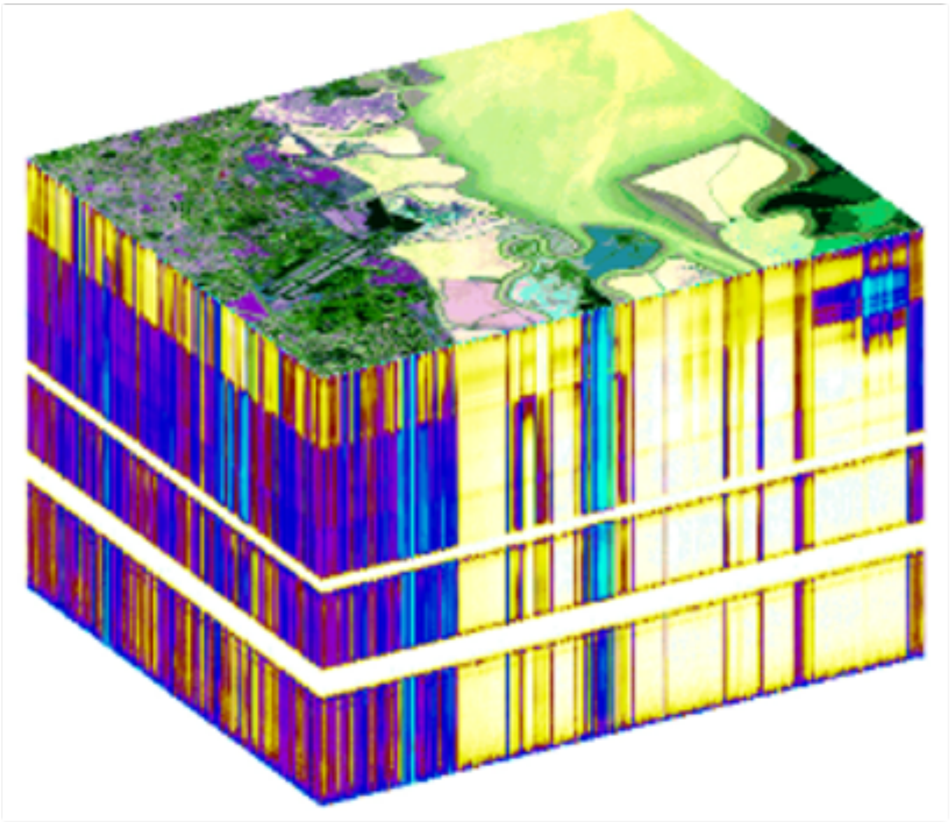
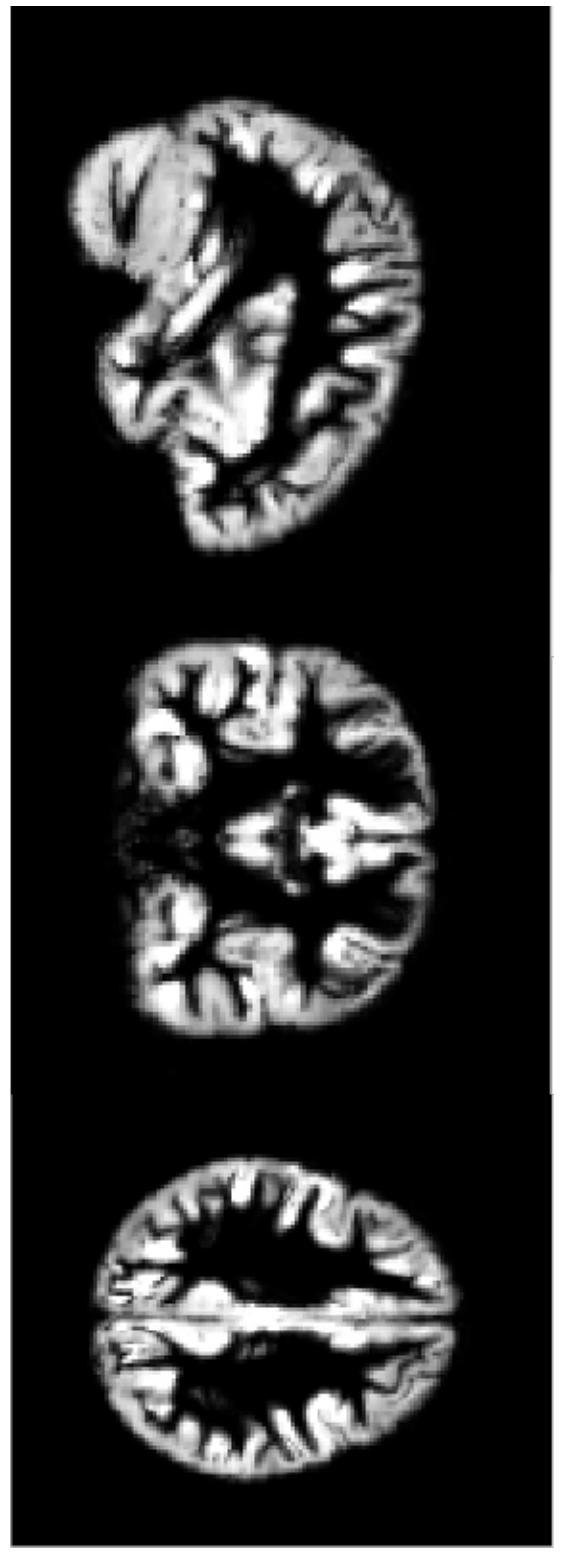
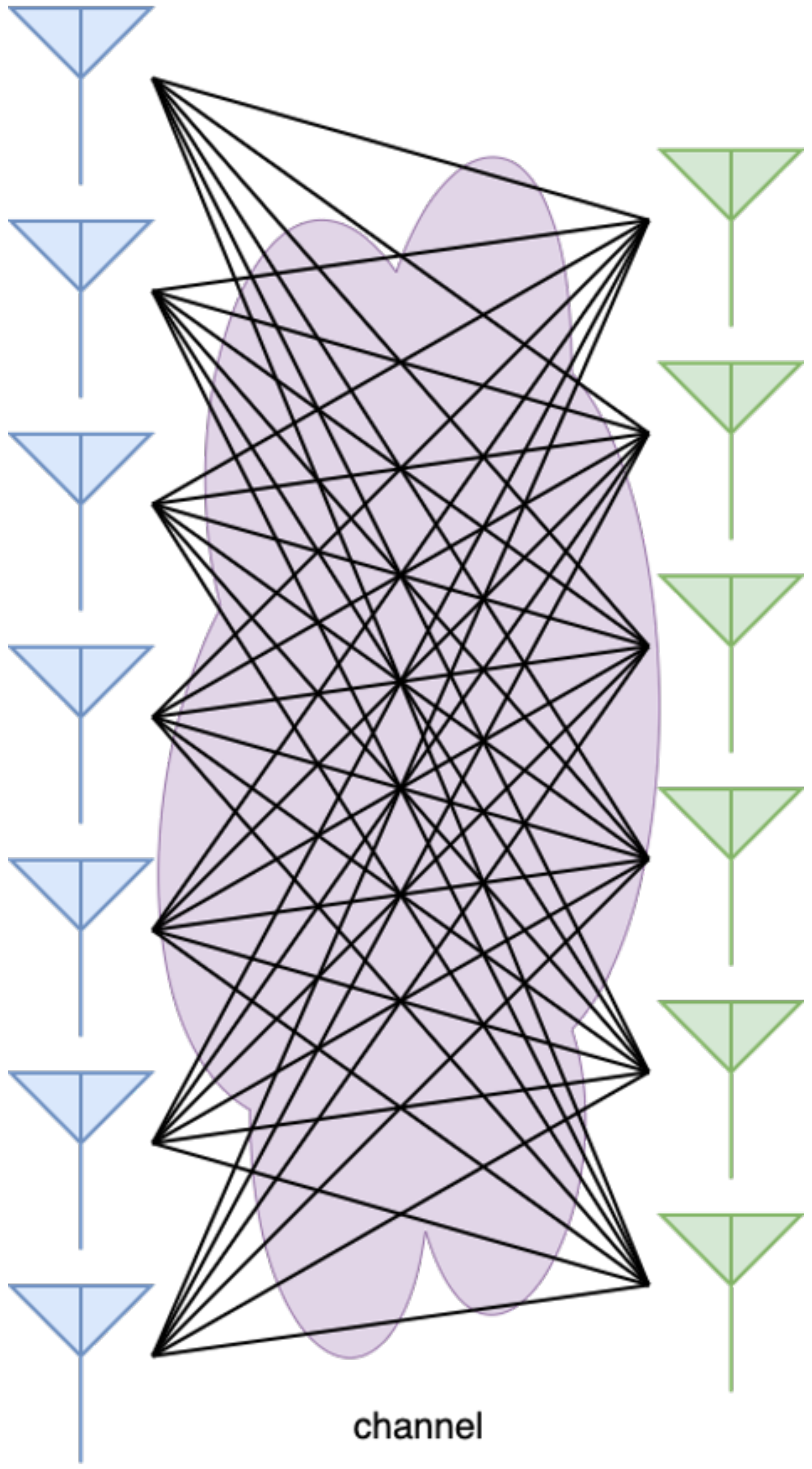
- Also quantum physics, chemometrics, numerical linear algebra, psychometrics, theoretical computer science…









What do we want to do with tensor data?

All the regular things we do with data…

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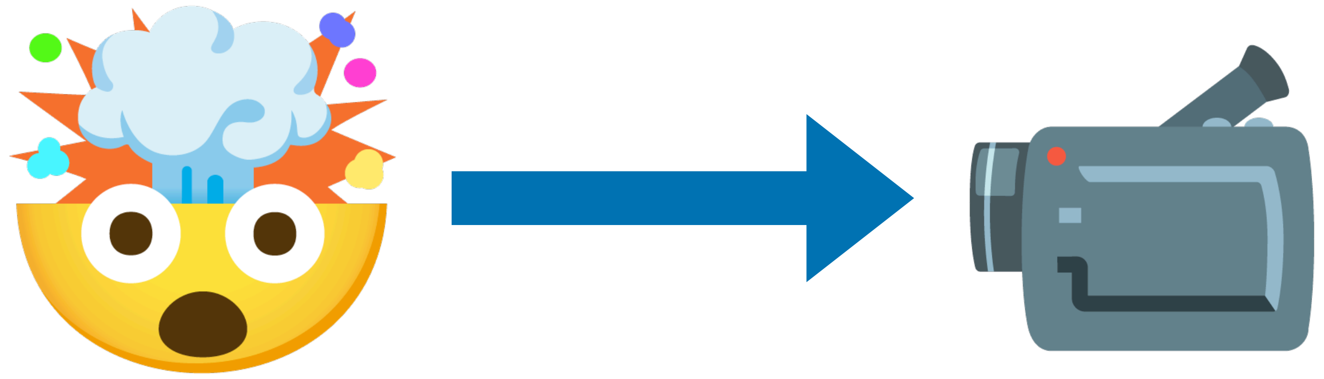

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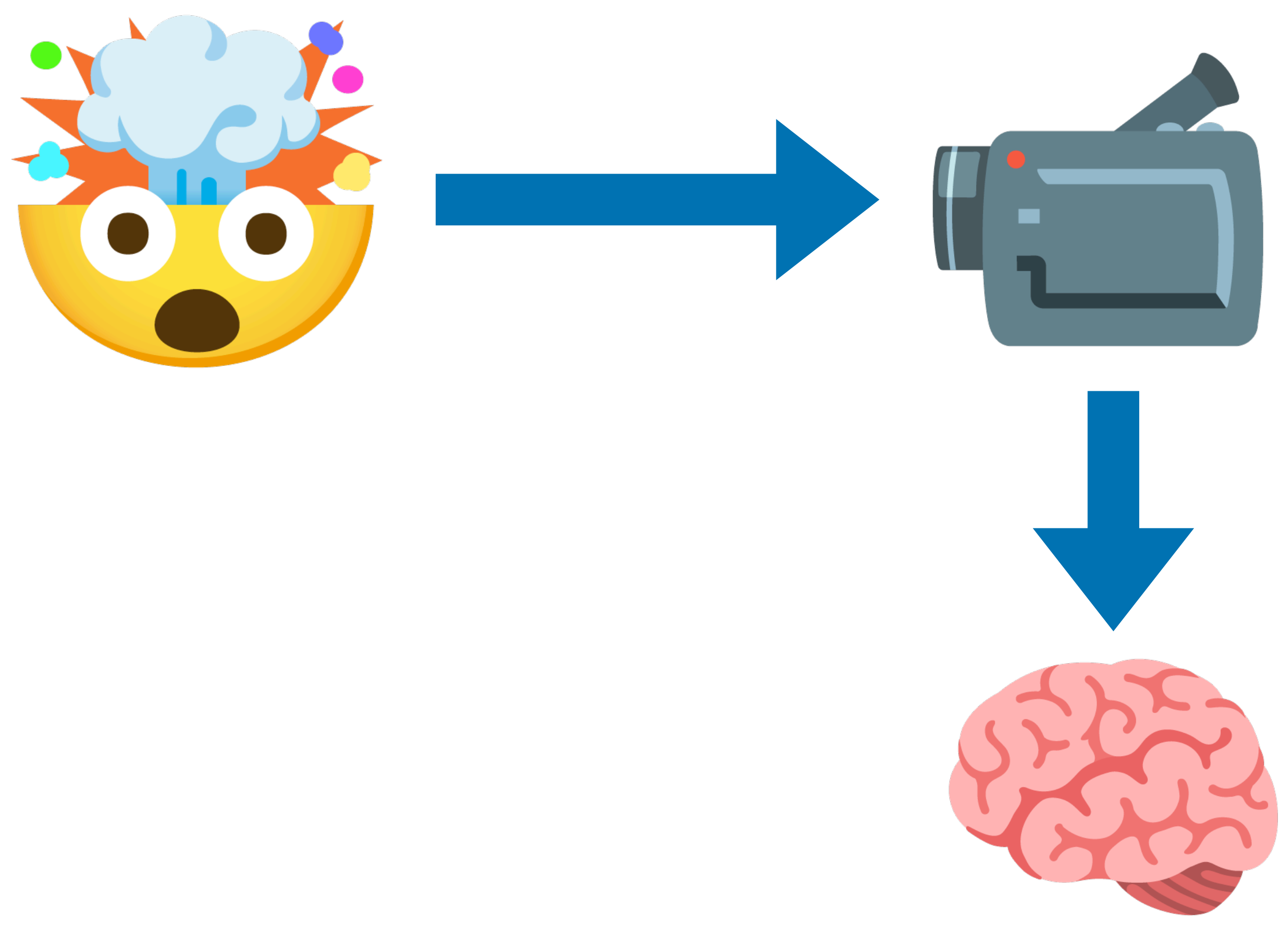

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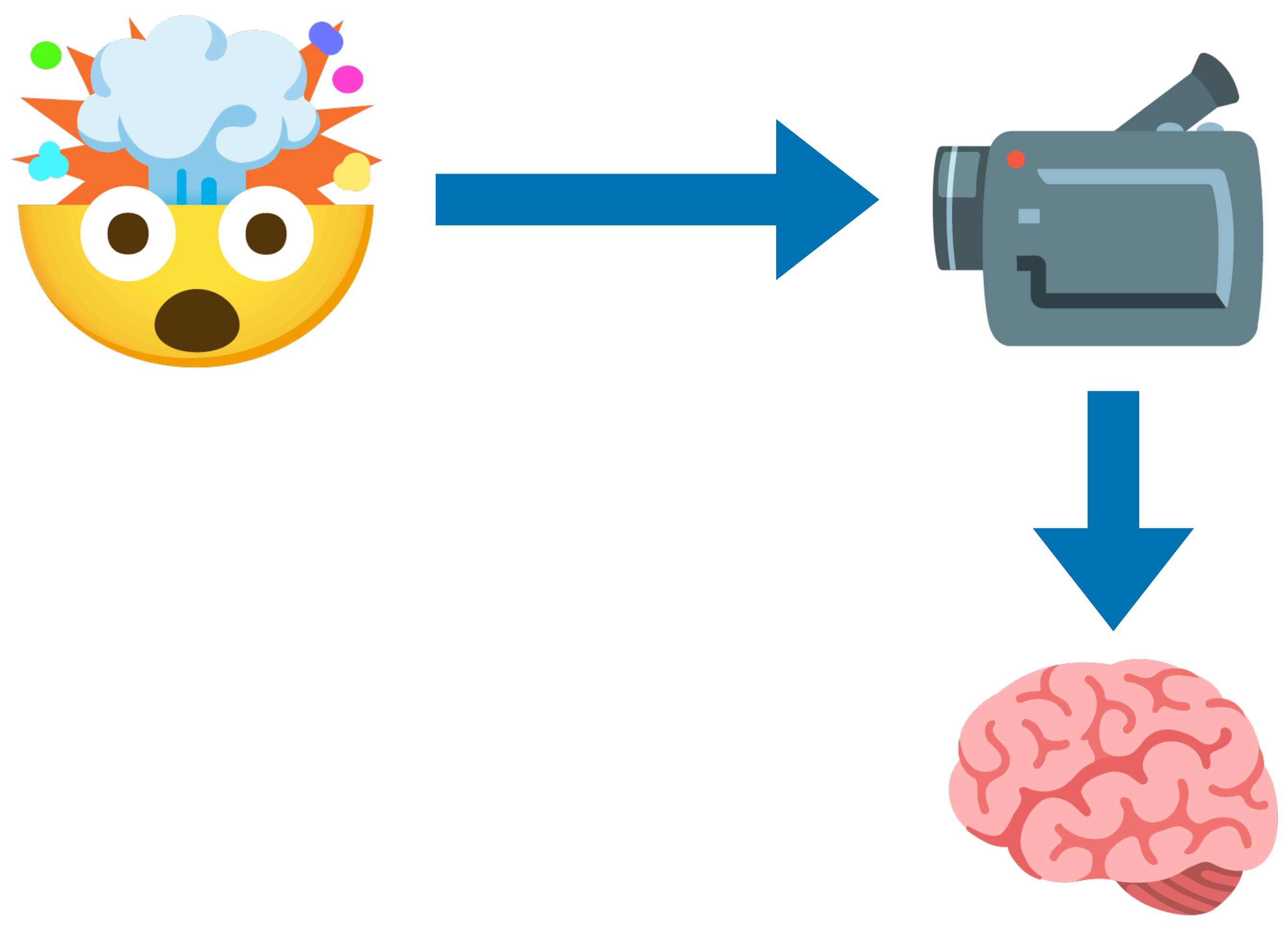

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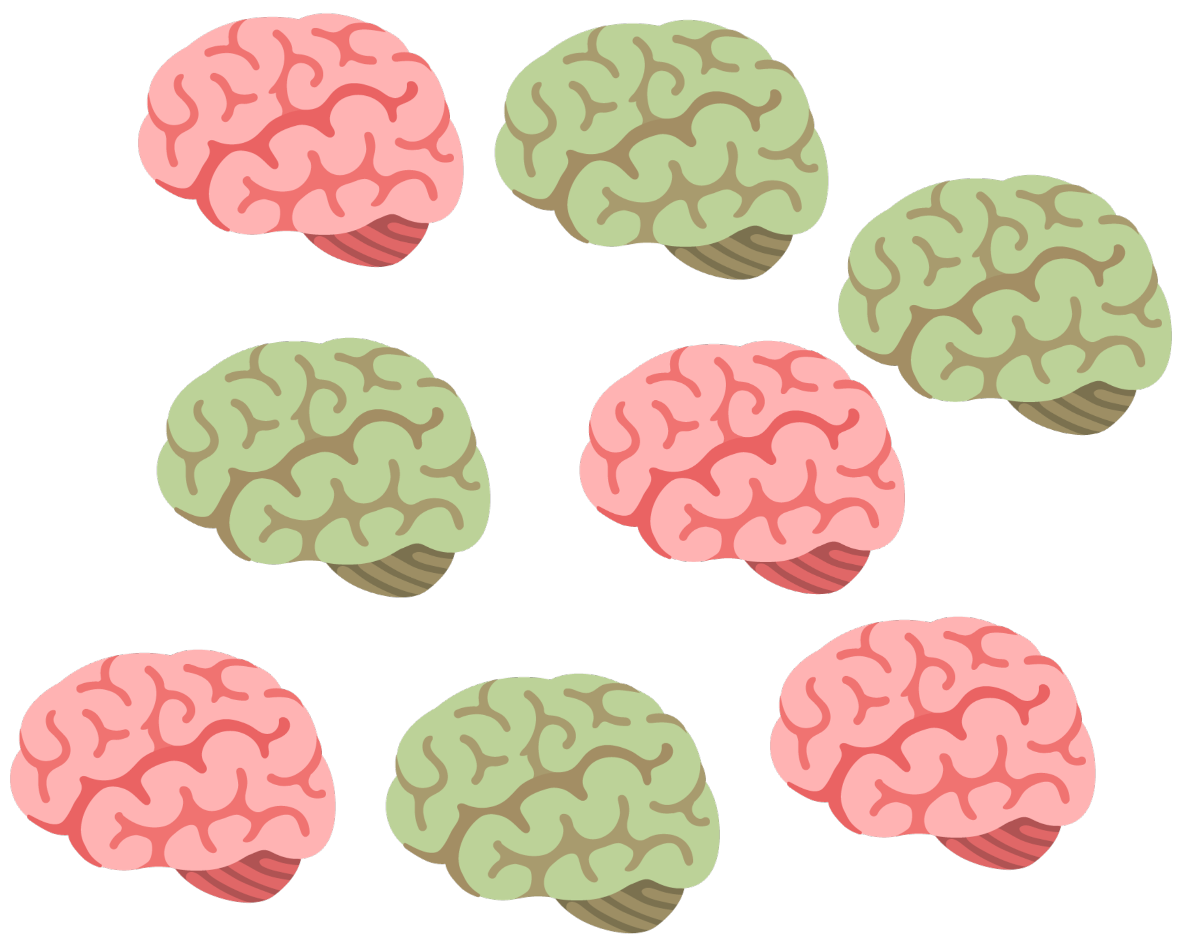

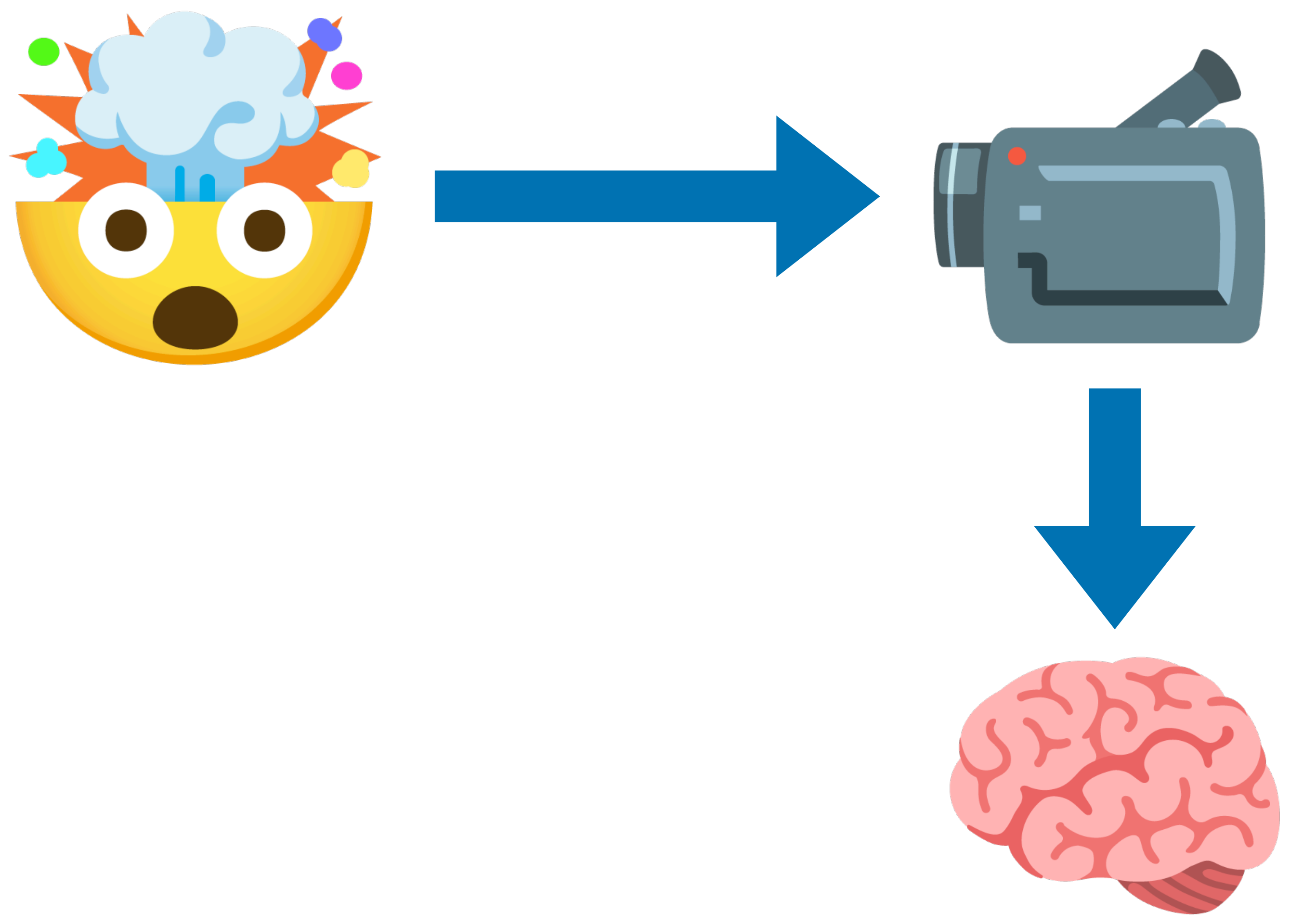

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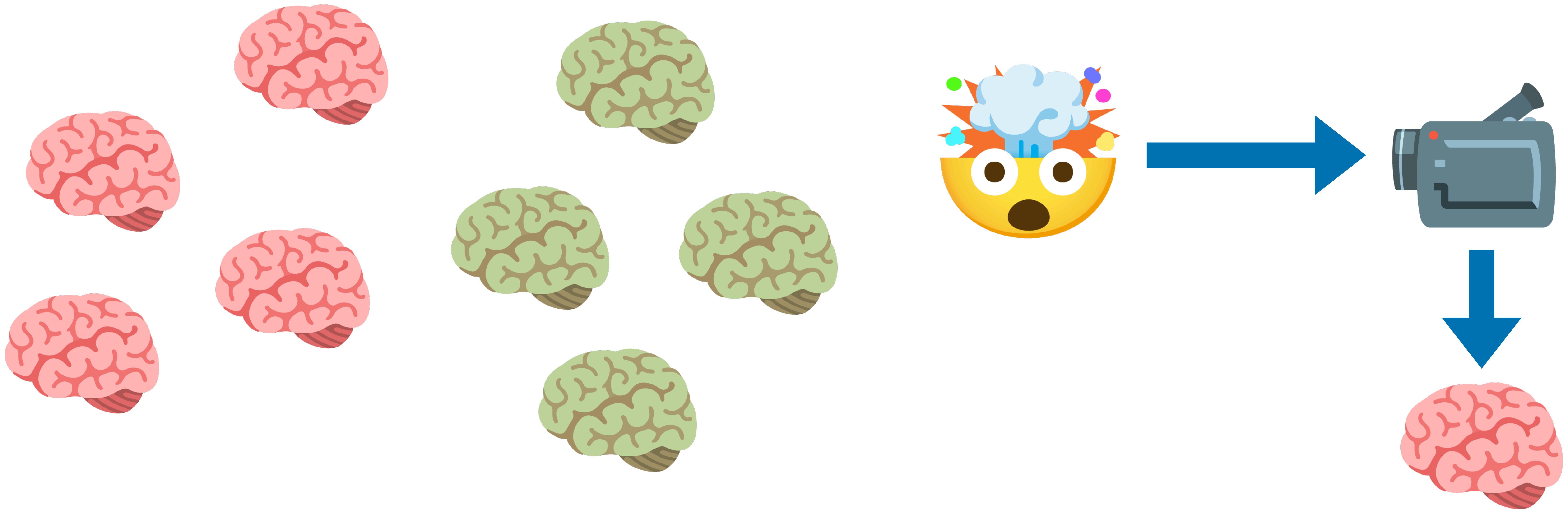

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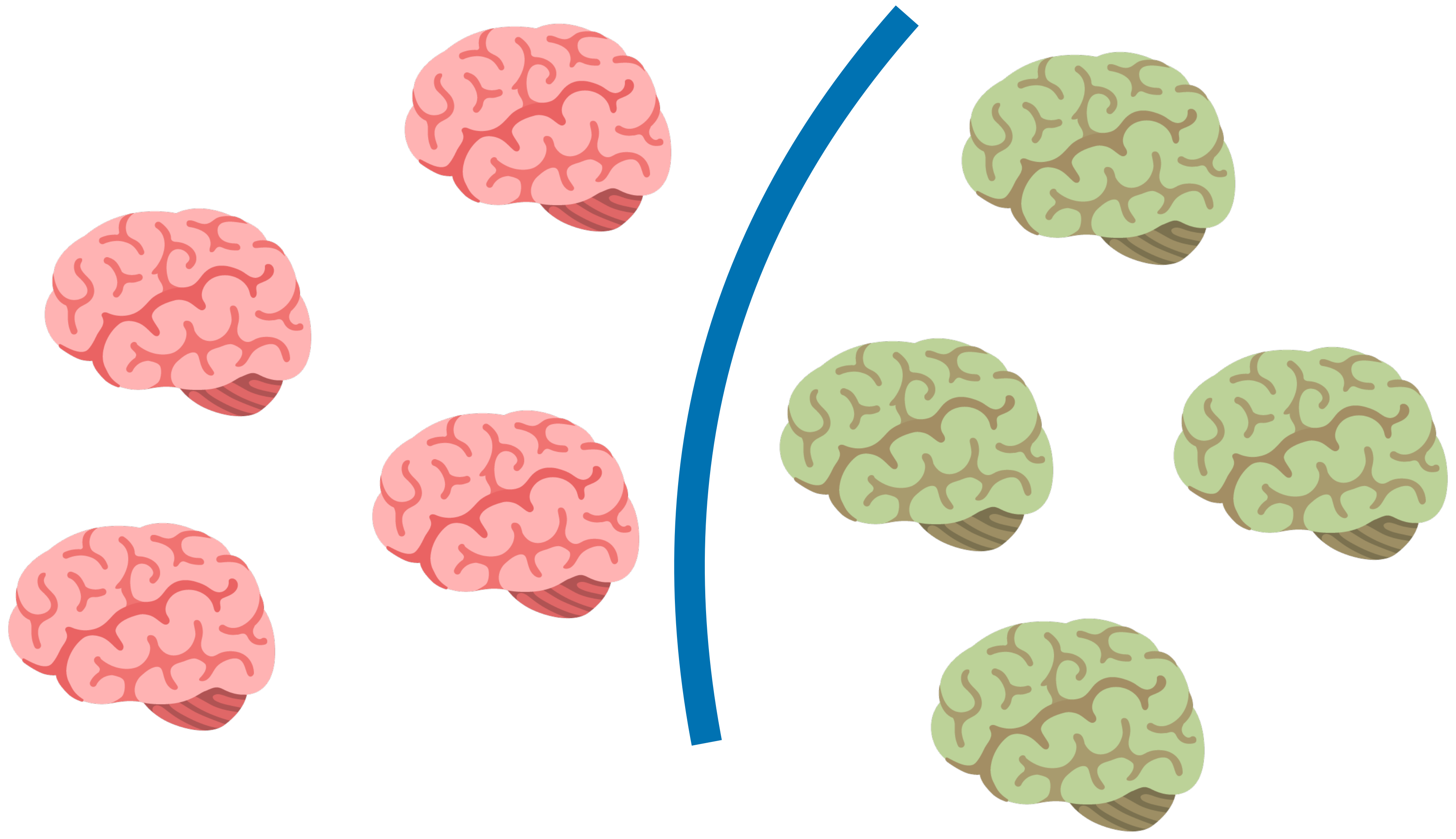

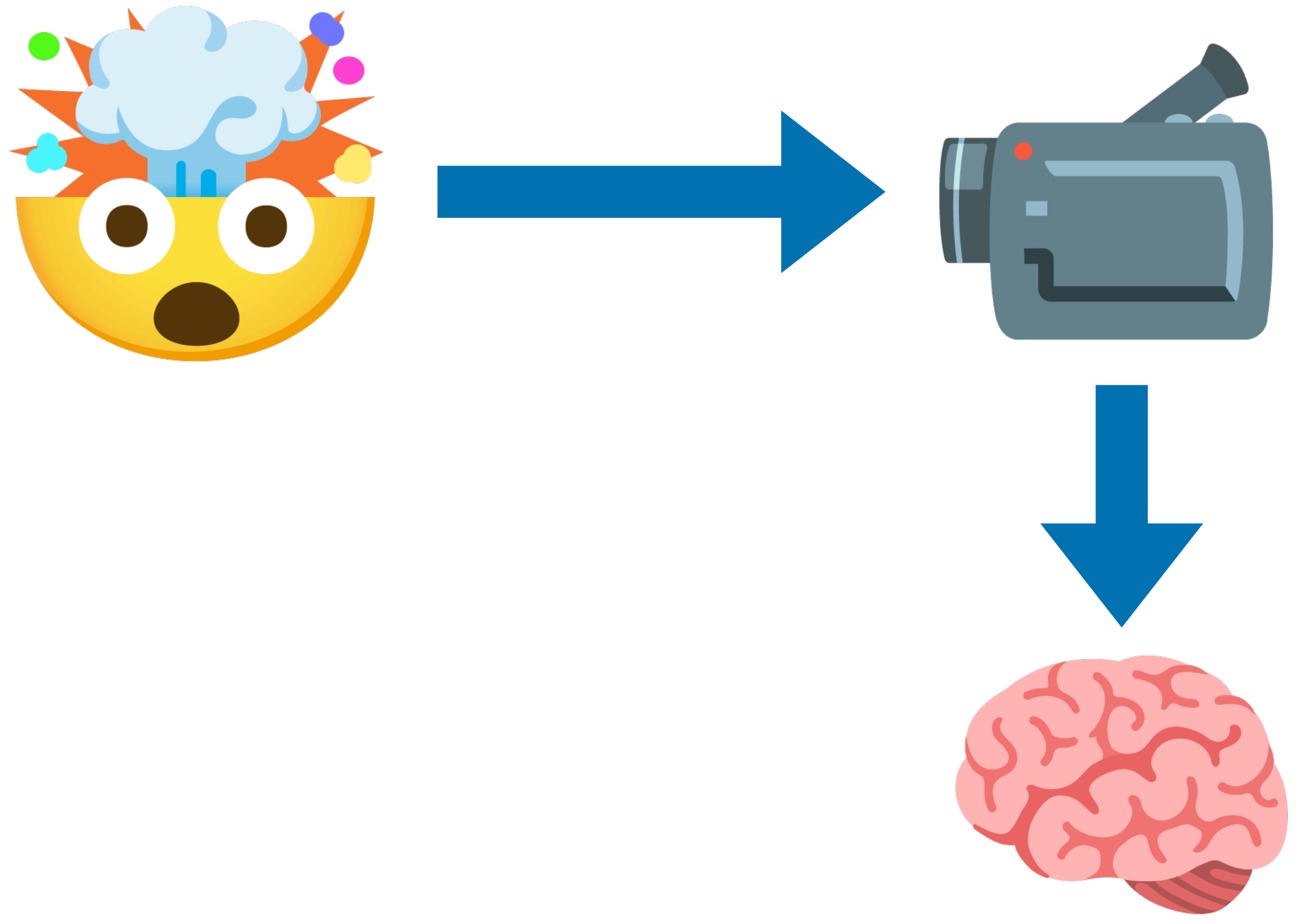

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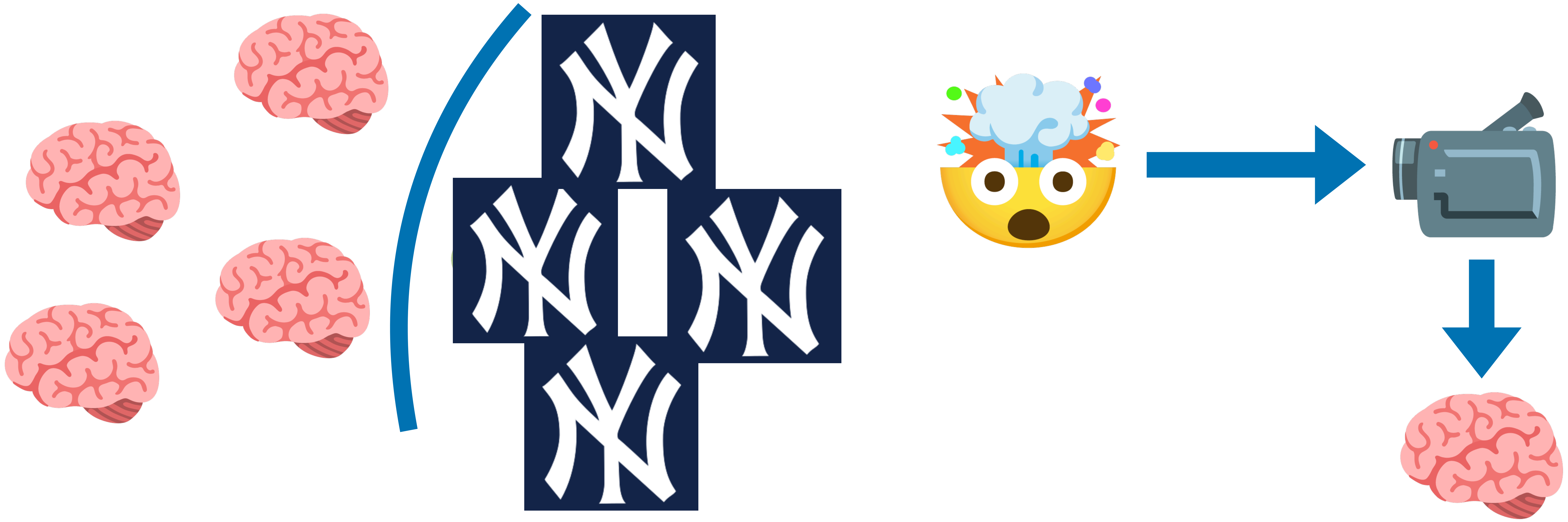

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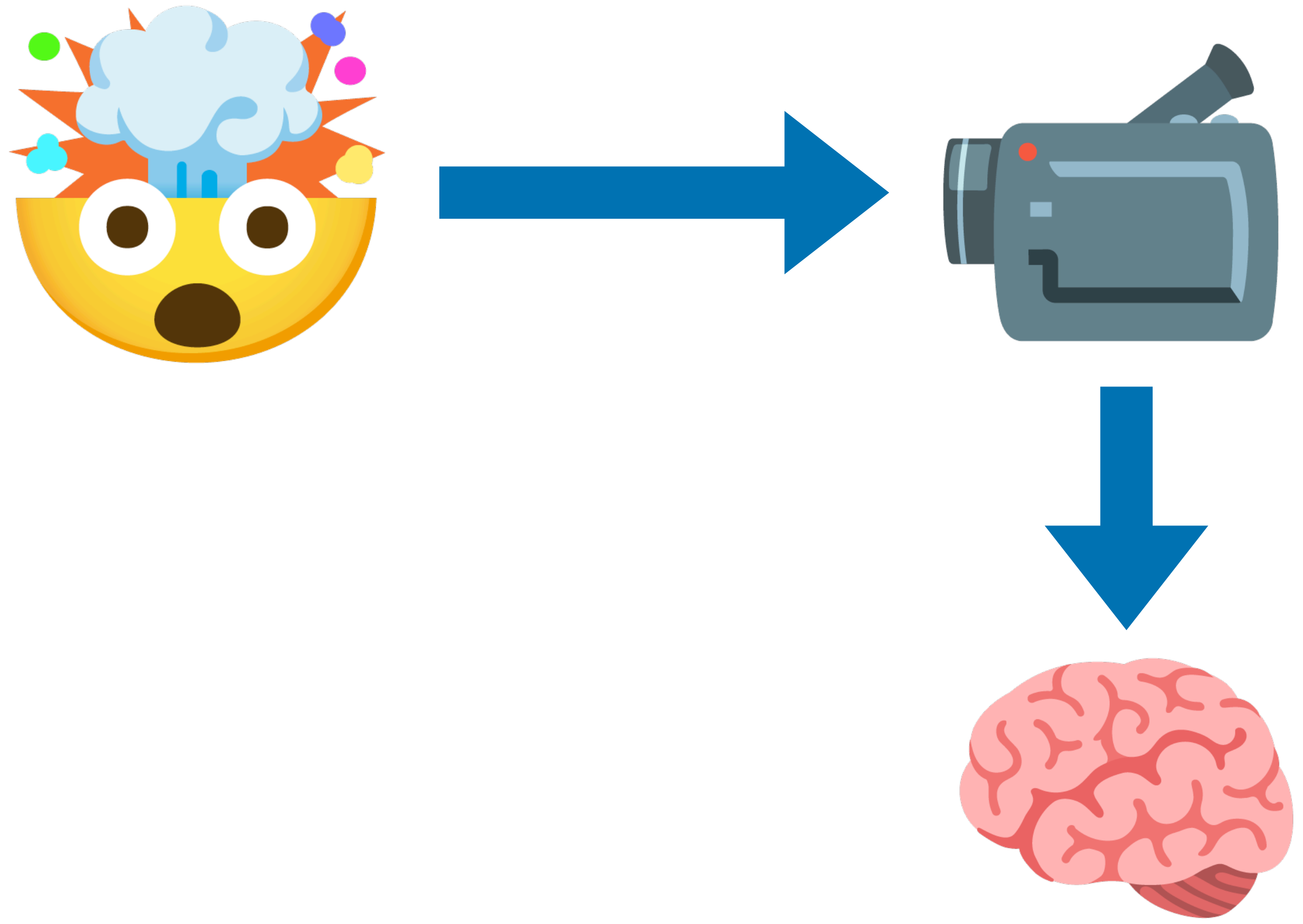

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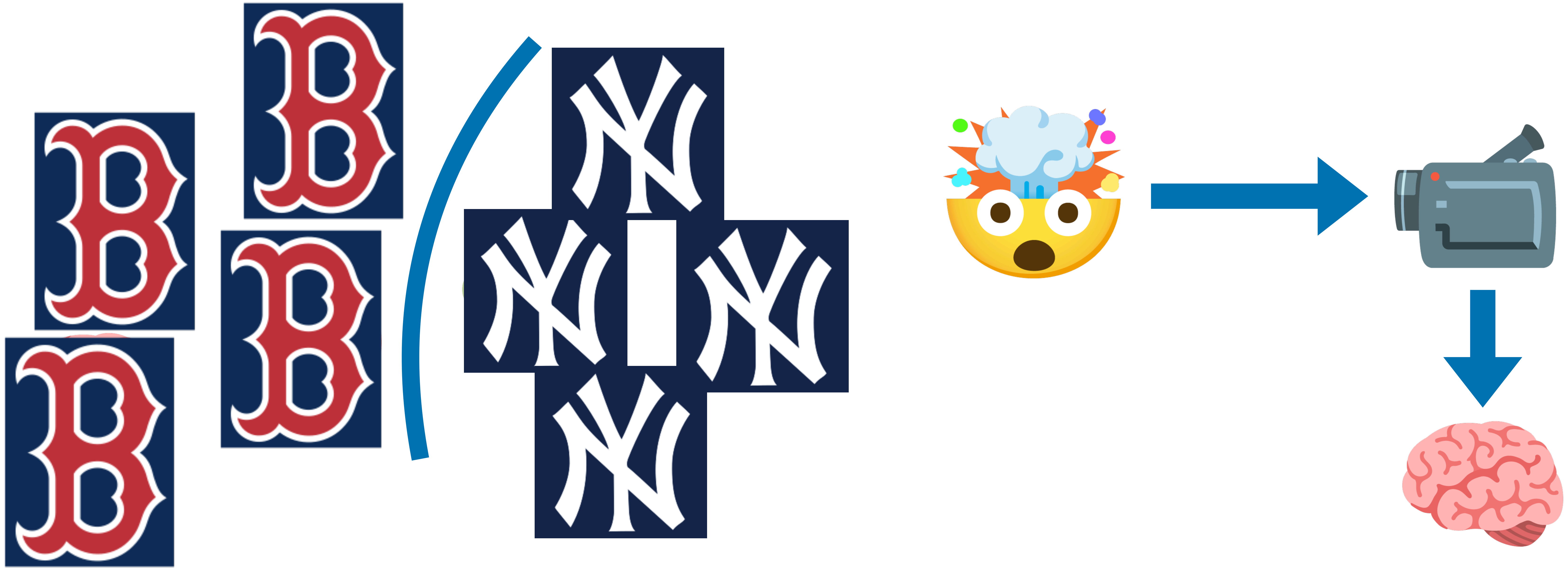

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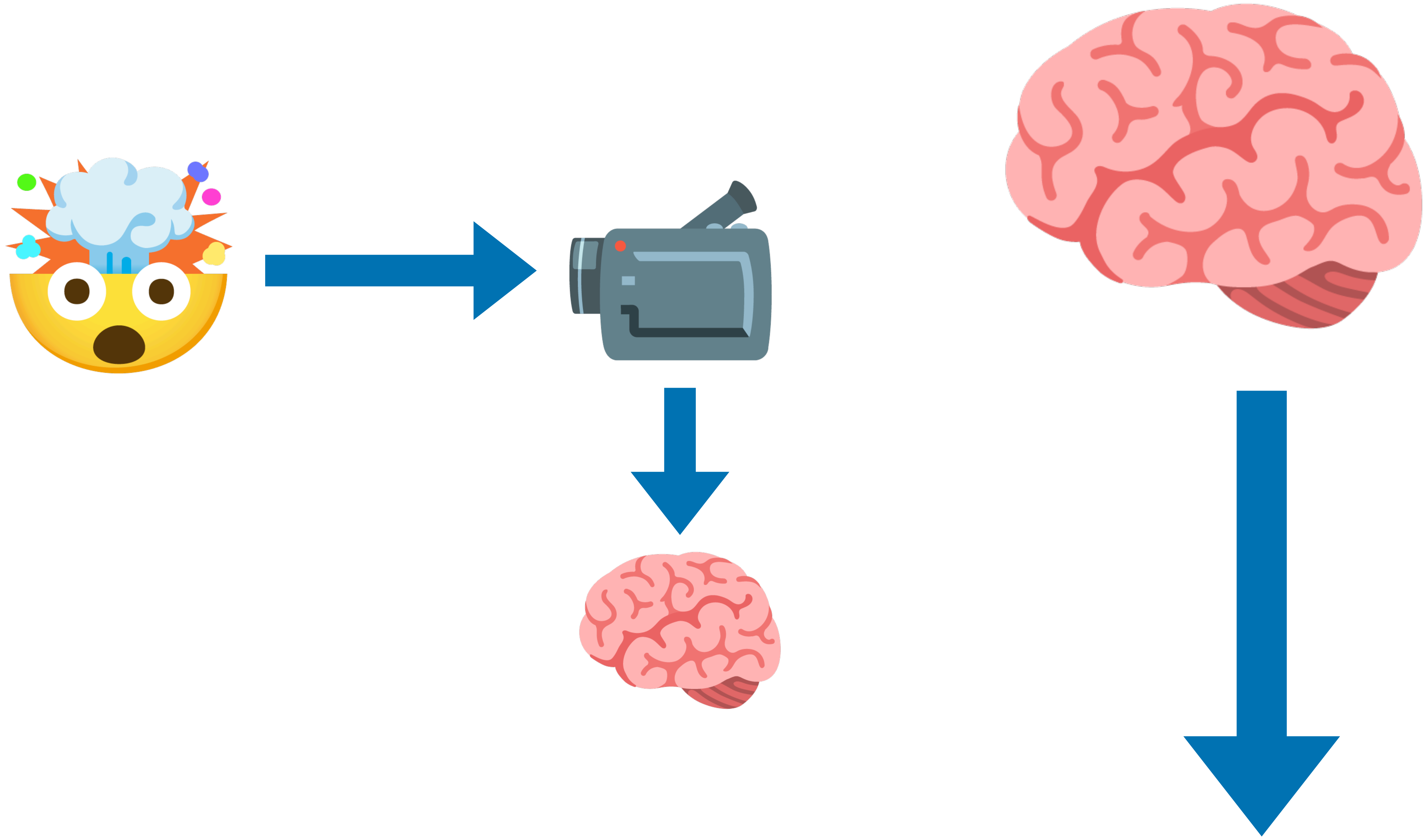

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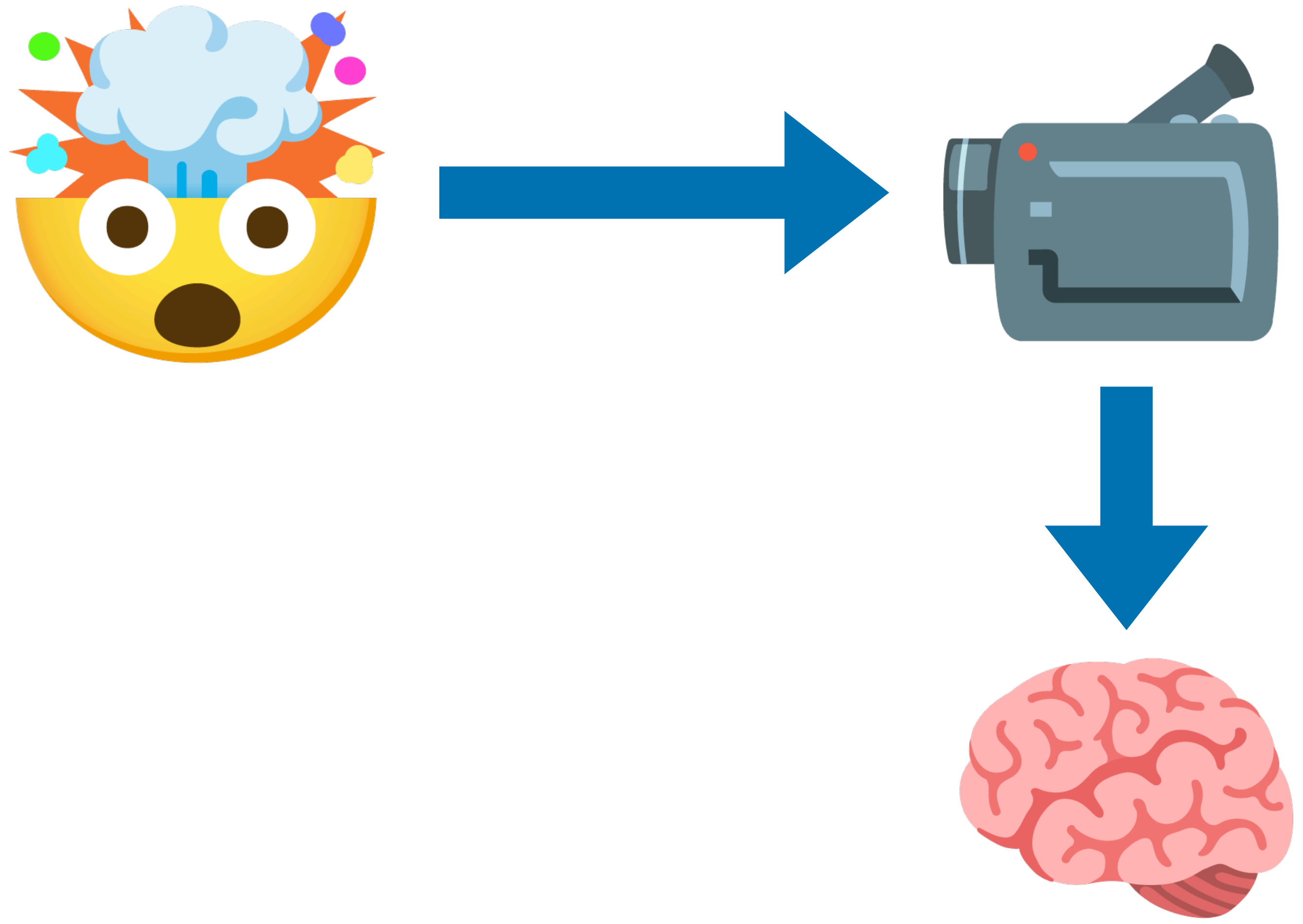

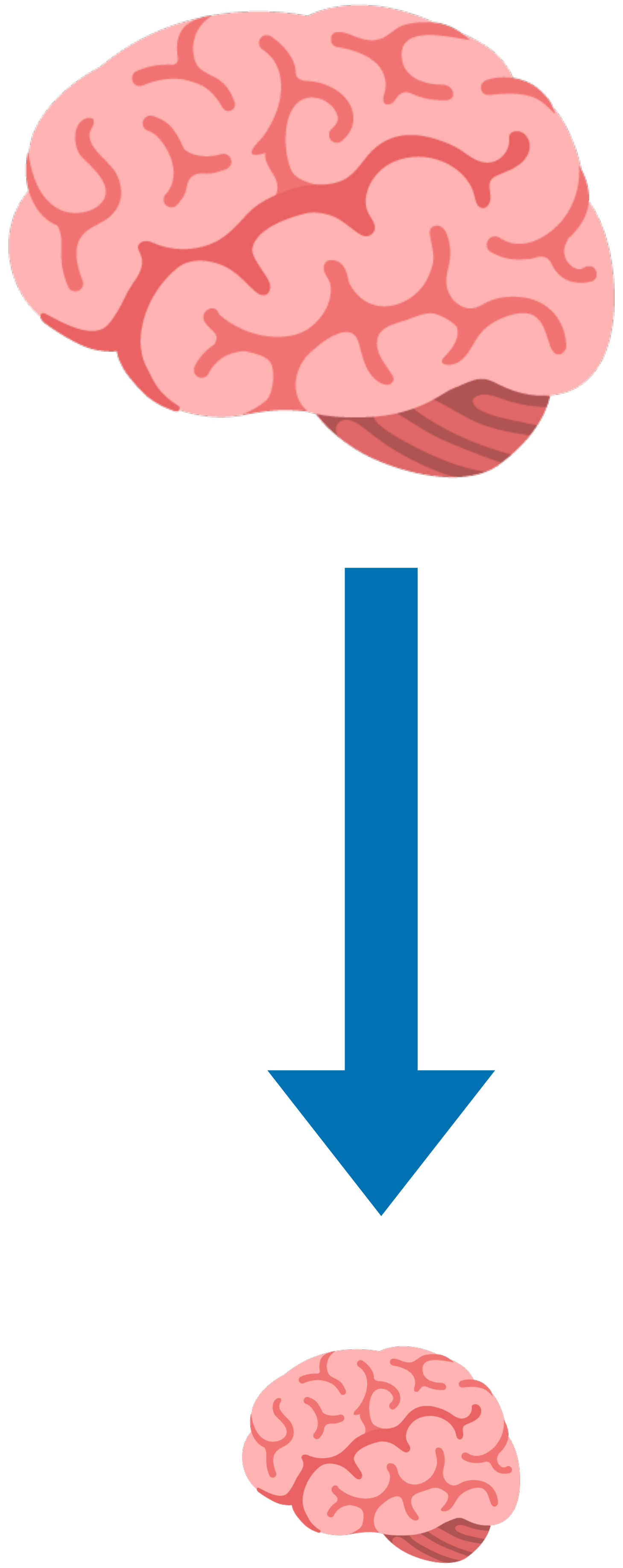

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Unsupervised learning with tensors

Example: dictionary learning and sparse representations

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Task: given a collection of tensors $\underline{\mathbf{Y}}_1, \underline{\mathbf{Y}}_2, \ldots, \underline{\mathbf{Y}}_n \in \mathbb{R}^{m_1 \times m_2 \times \cdots \times m_K}$, find a *dictionary* $\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \ldots, \underline{\mathbf{d}}_p$ such that

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Application: processing or storing hyperspectral images acquired from a drone.

Supervised learning with tensors

Exampled: regression with tensor-valued covariates

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Task: given a collection of tensor-scalar pairs $\{(\underline{\mathbf{X}}_i, y_i)\} \subset \mathbb{R}^{m_1 \times m_2 \times \cdots \times m_K} \times \mathbb{R}$, find a *regression tensor* $\underline{\mathbf{B}}$ such that

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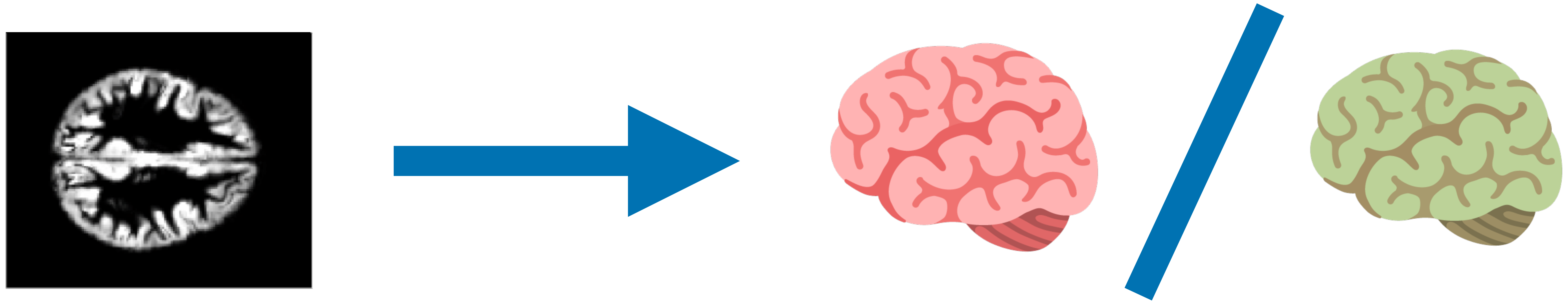
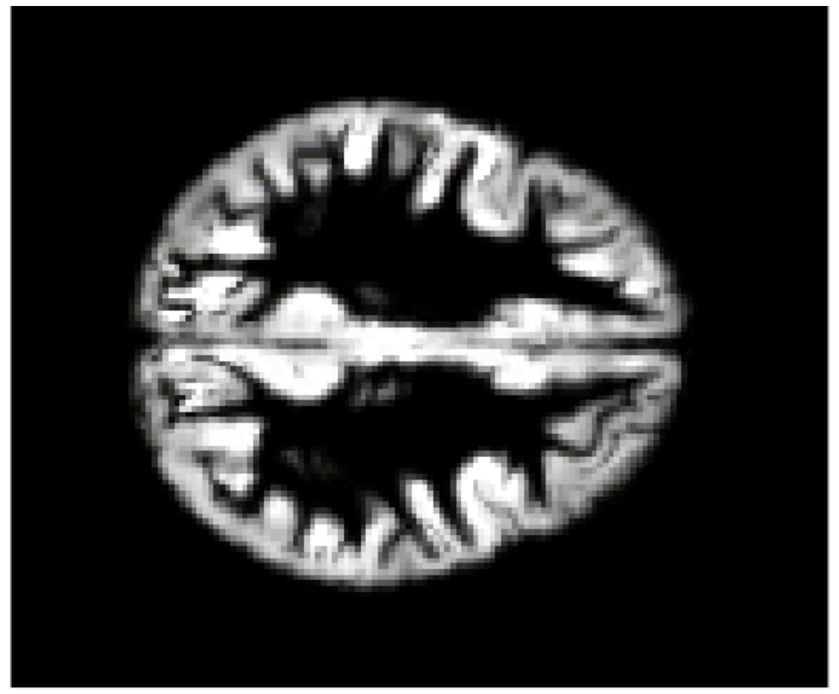
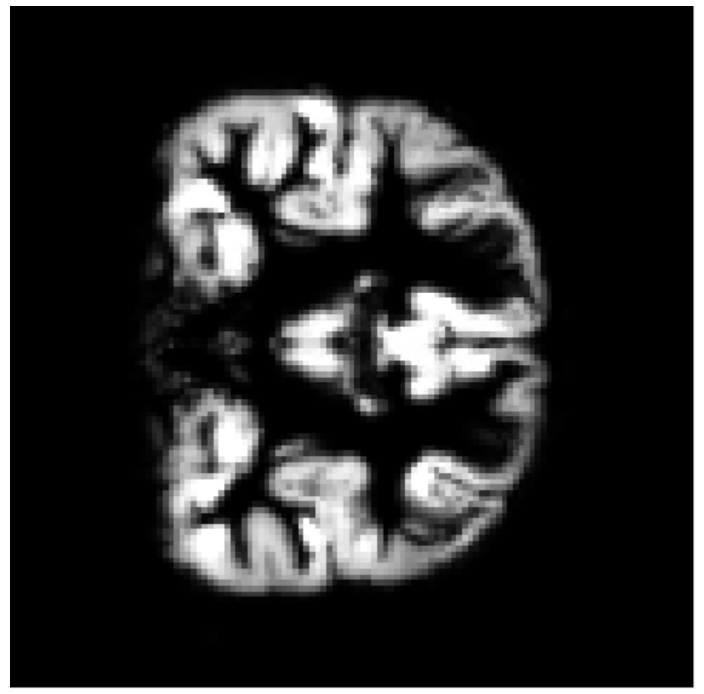


Why not use large "foundation" models?

For many applications, data is high-dimensional and expensive



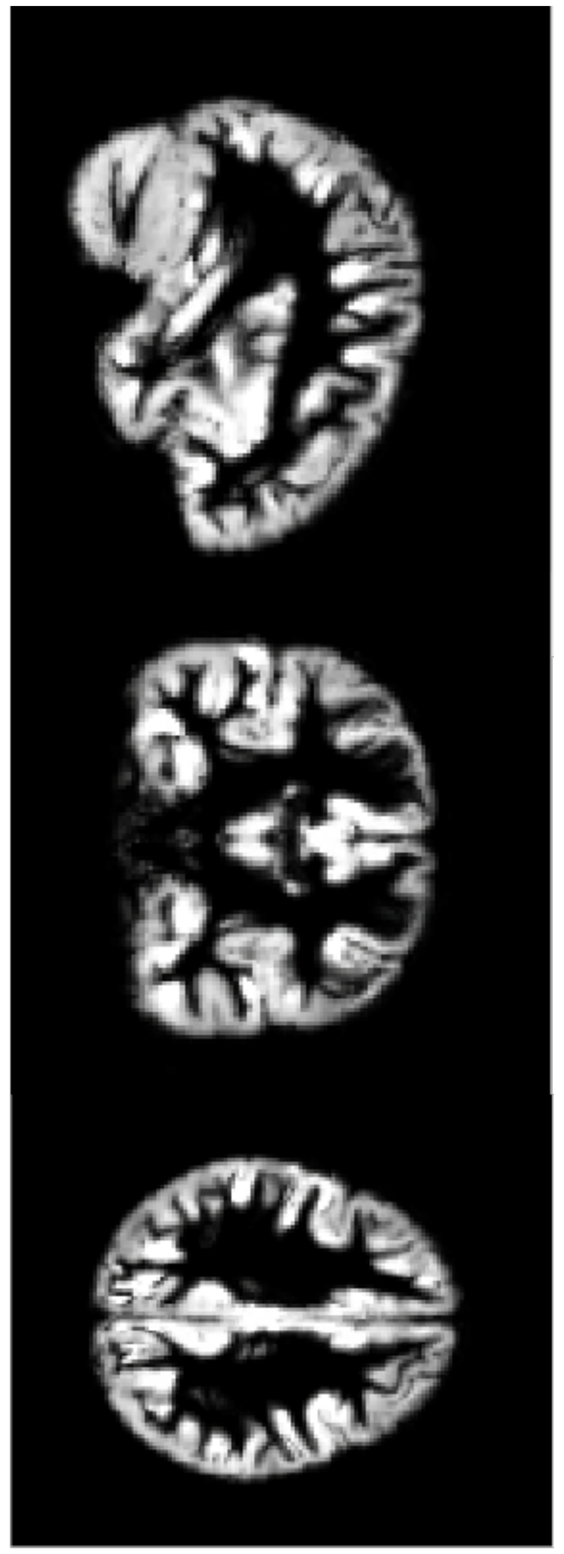


Example: ADHD-200 sample aggregates 8 international imaging sites (US, Netherlands, China) with fMRI images of children's and adolescents' brains.

- fMRI data: 121 x 145 x 121 tensor
- After vectorizing: 2,122,945 dimensional vector
- Sample size: 959 total images

A baseline approach: reuse existing tools

We can always use `reshape( )`



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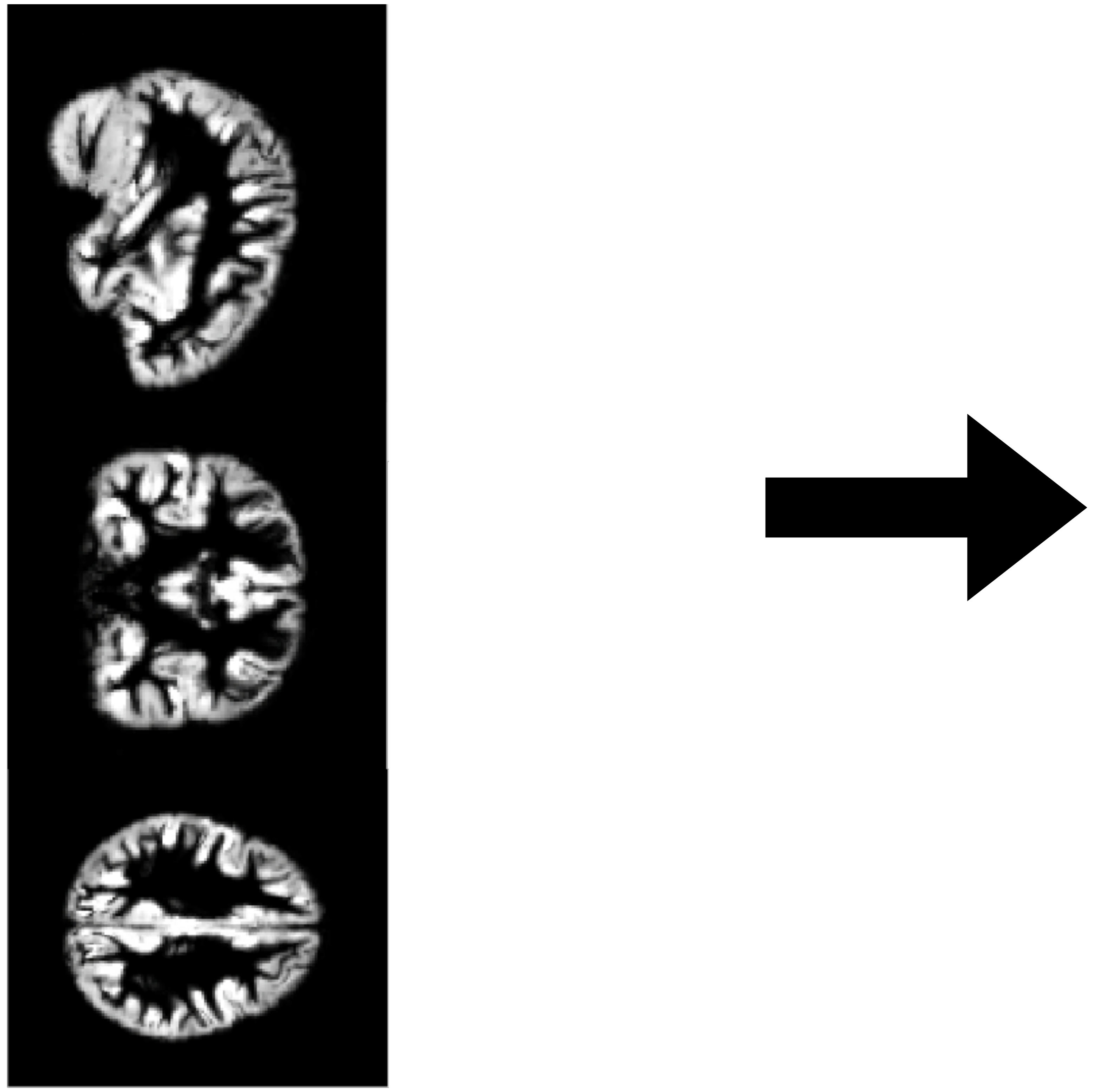

$m_1 \times m_2 \times m_3$

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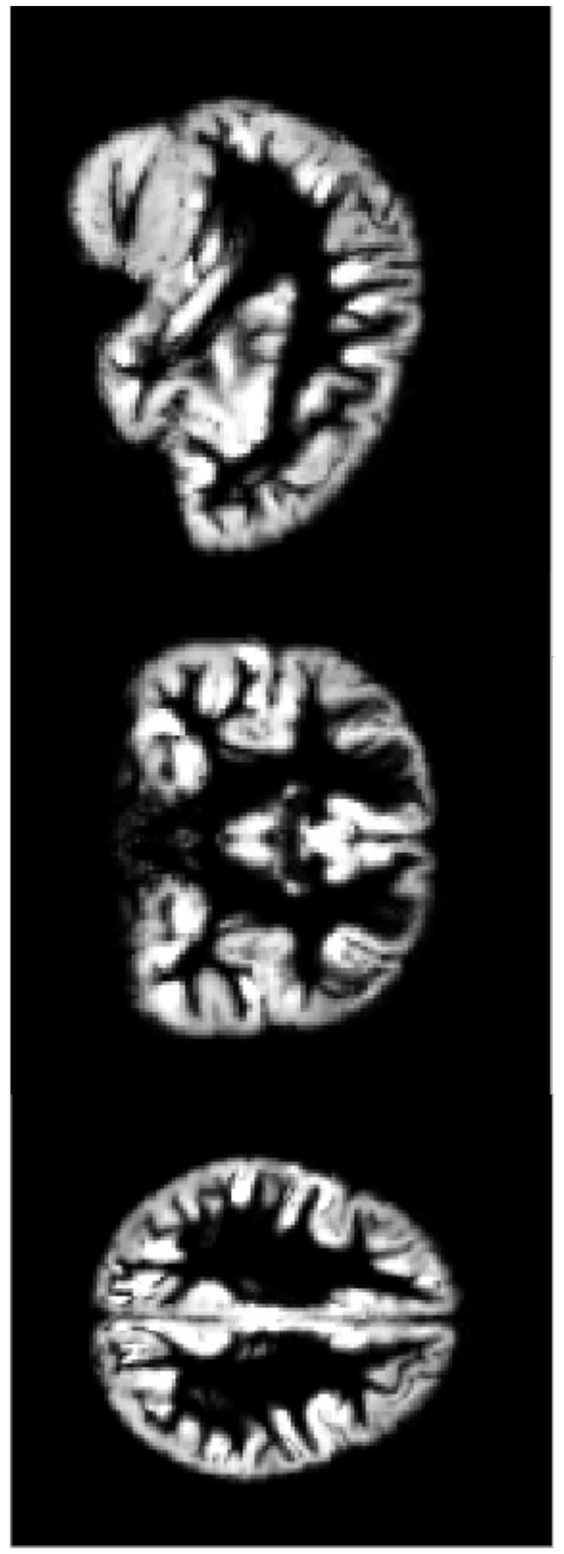

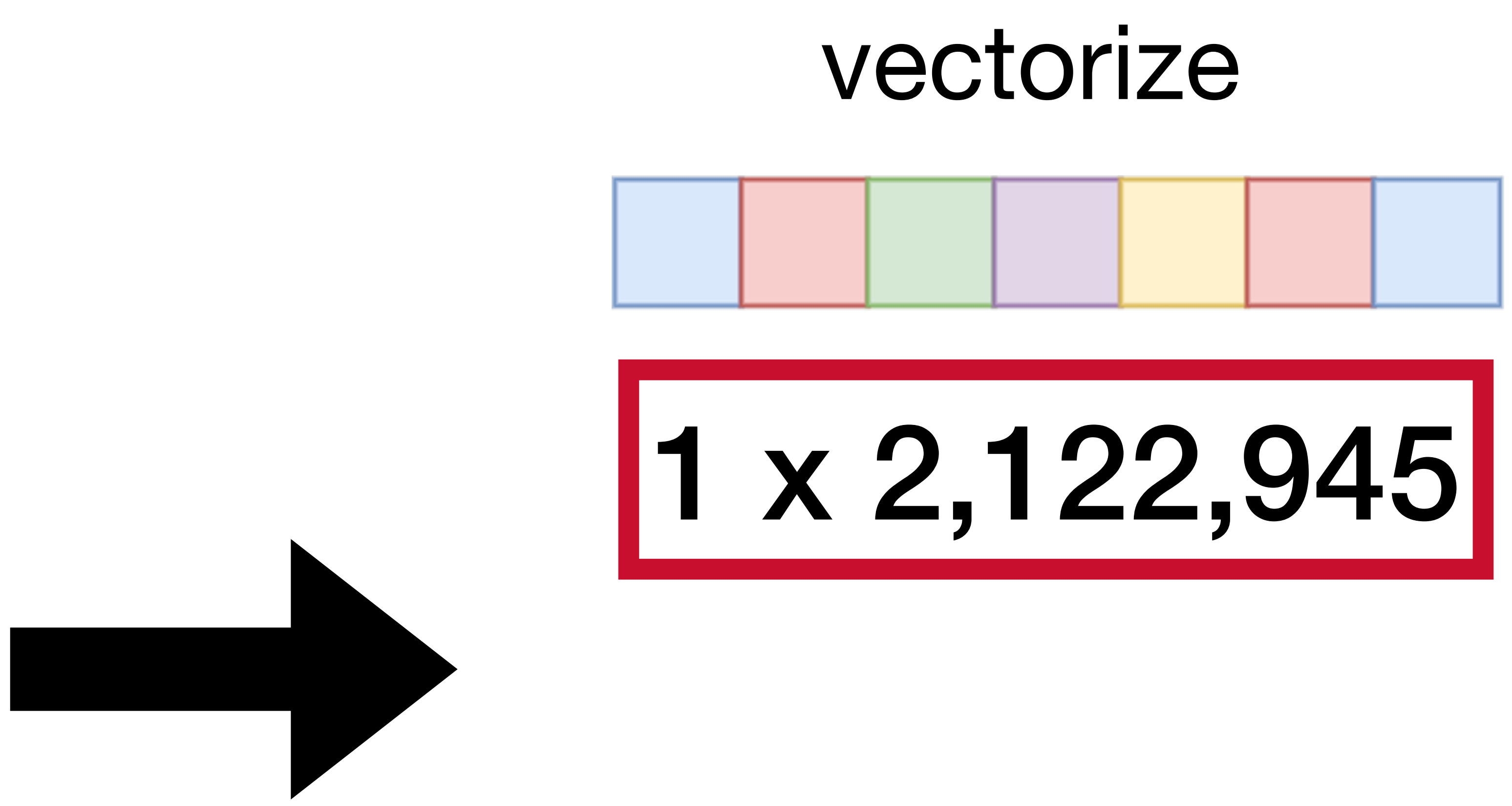

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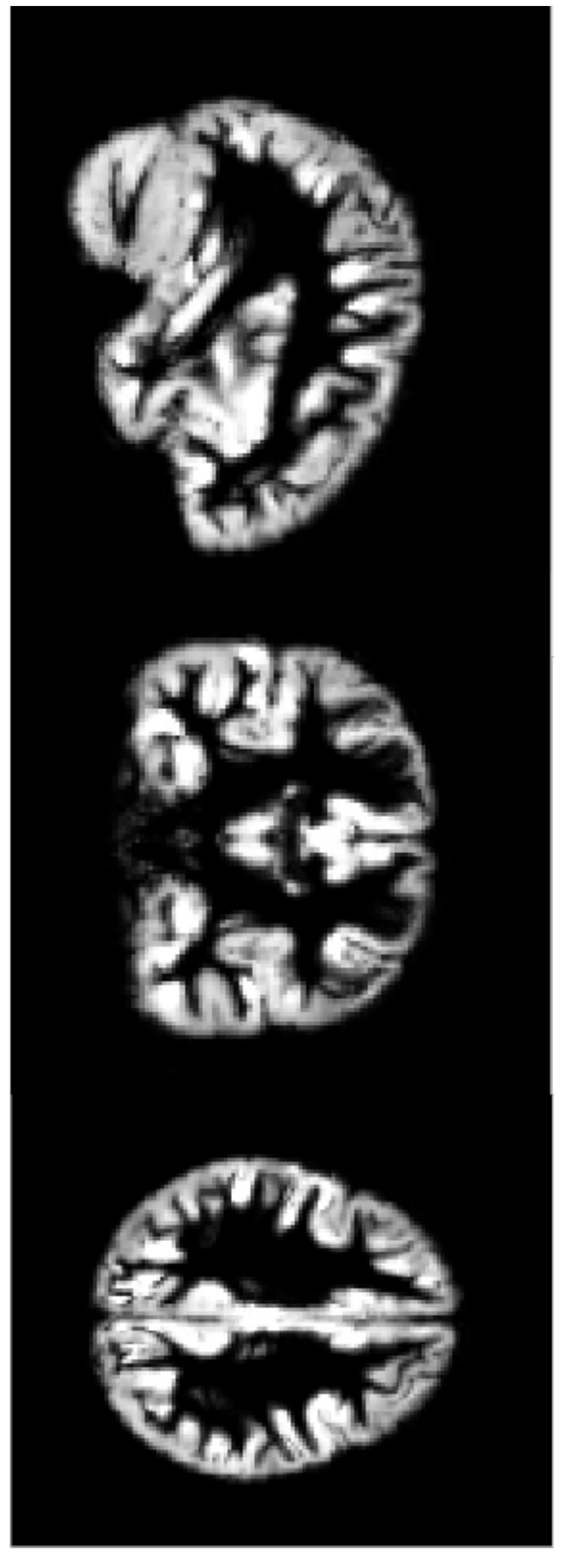

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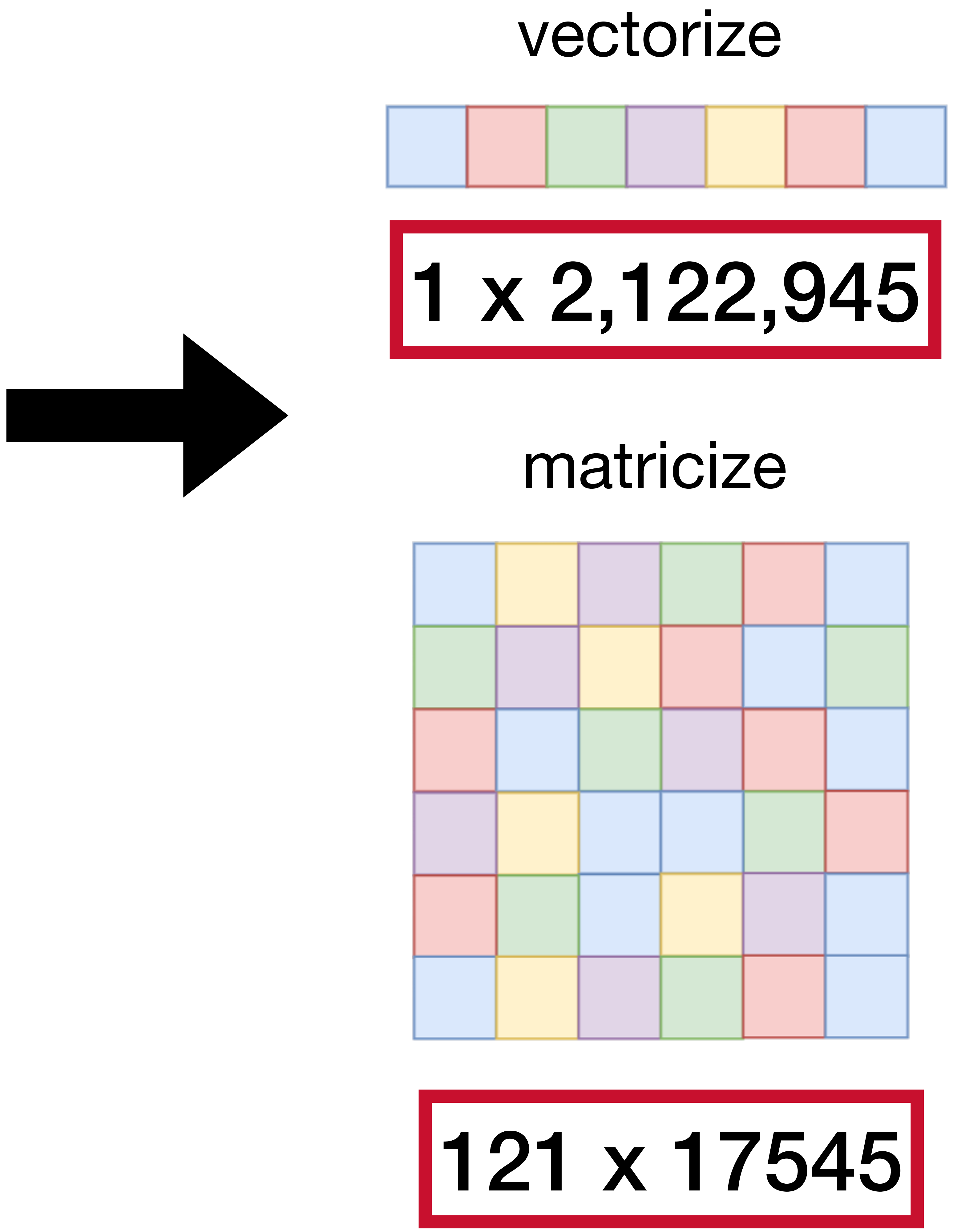

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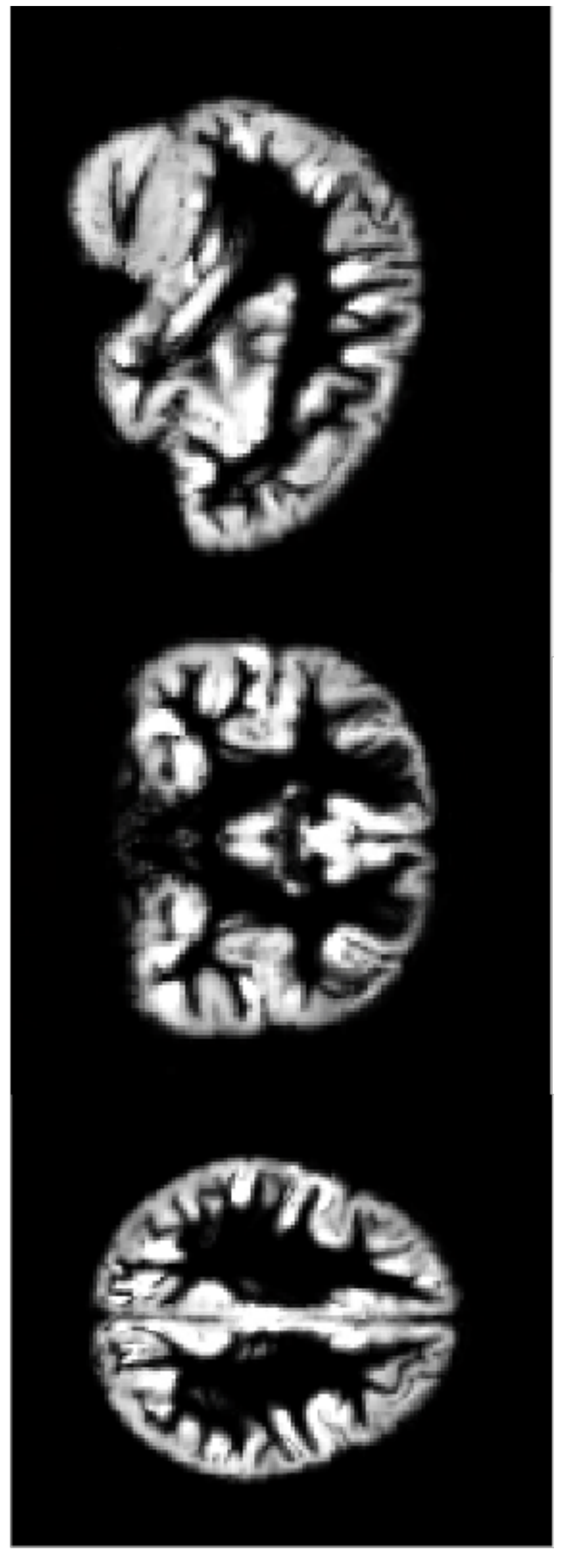


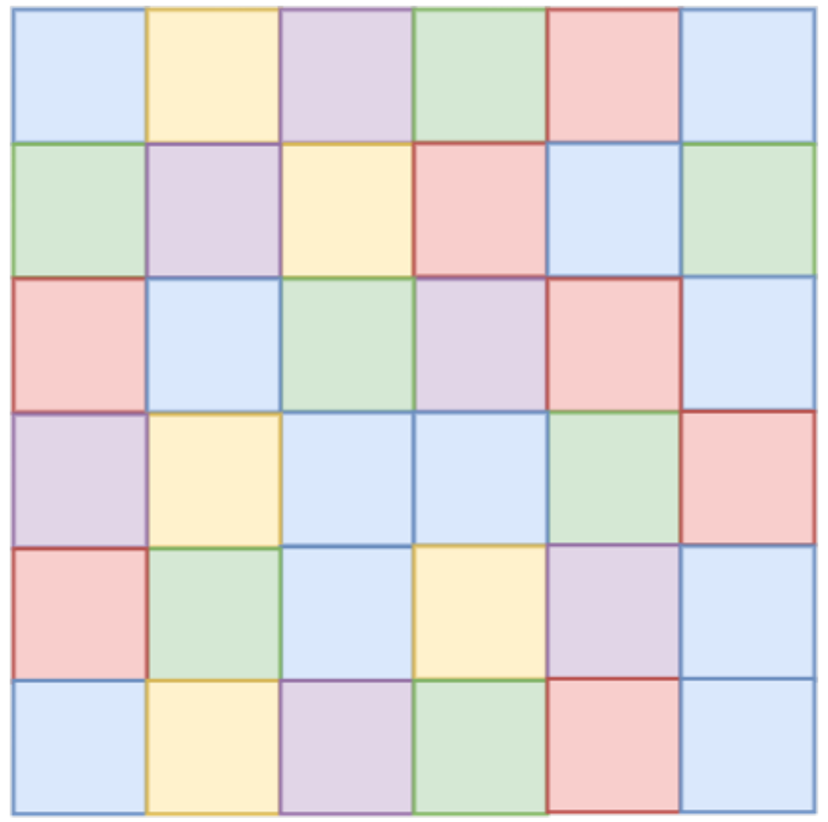
$m_1 \times m_2 \times m_3$

121 x 145 x 121

vectorize

1 x 2,122,945

matricize



121 x 17545

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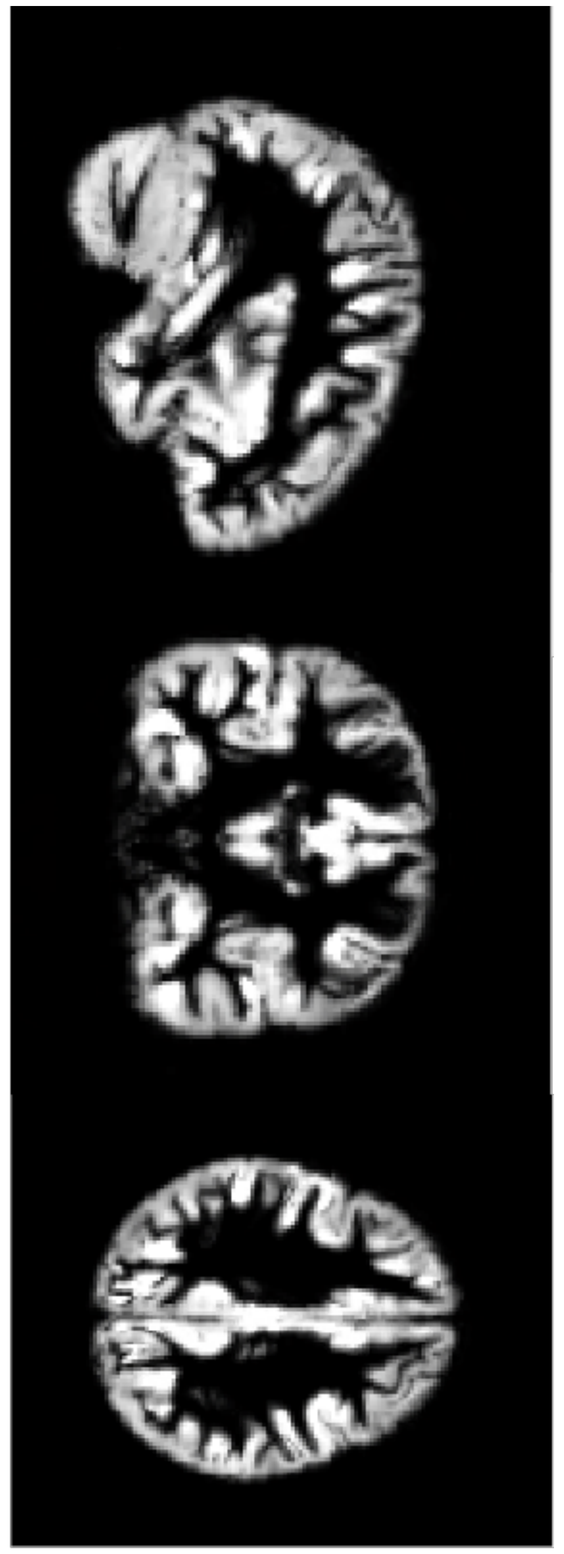

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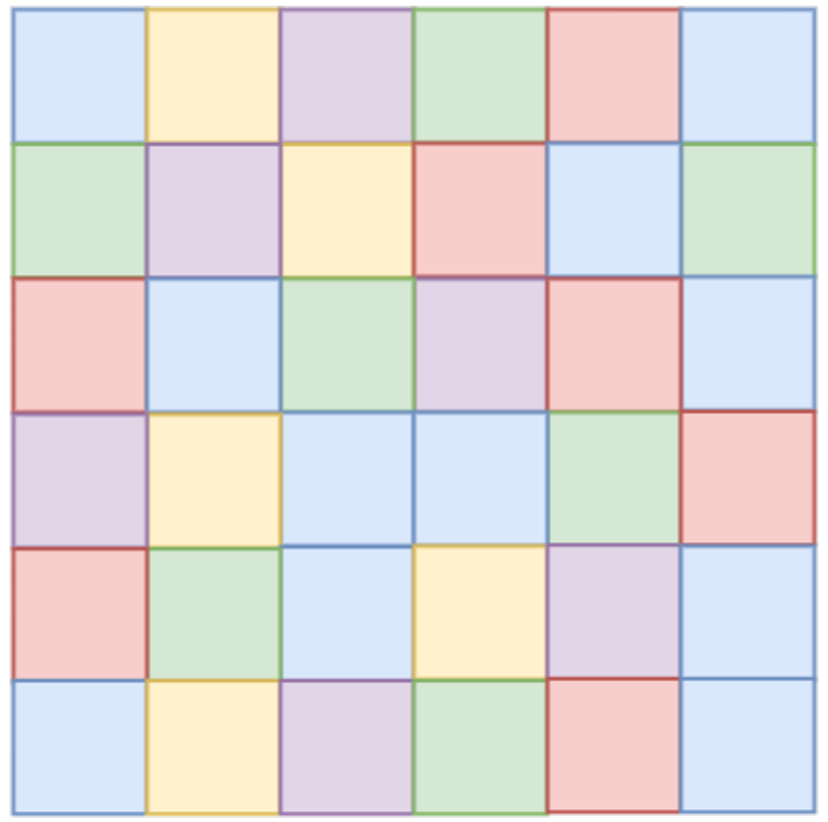

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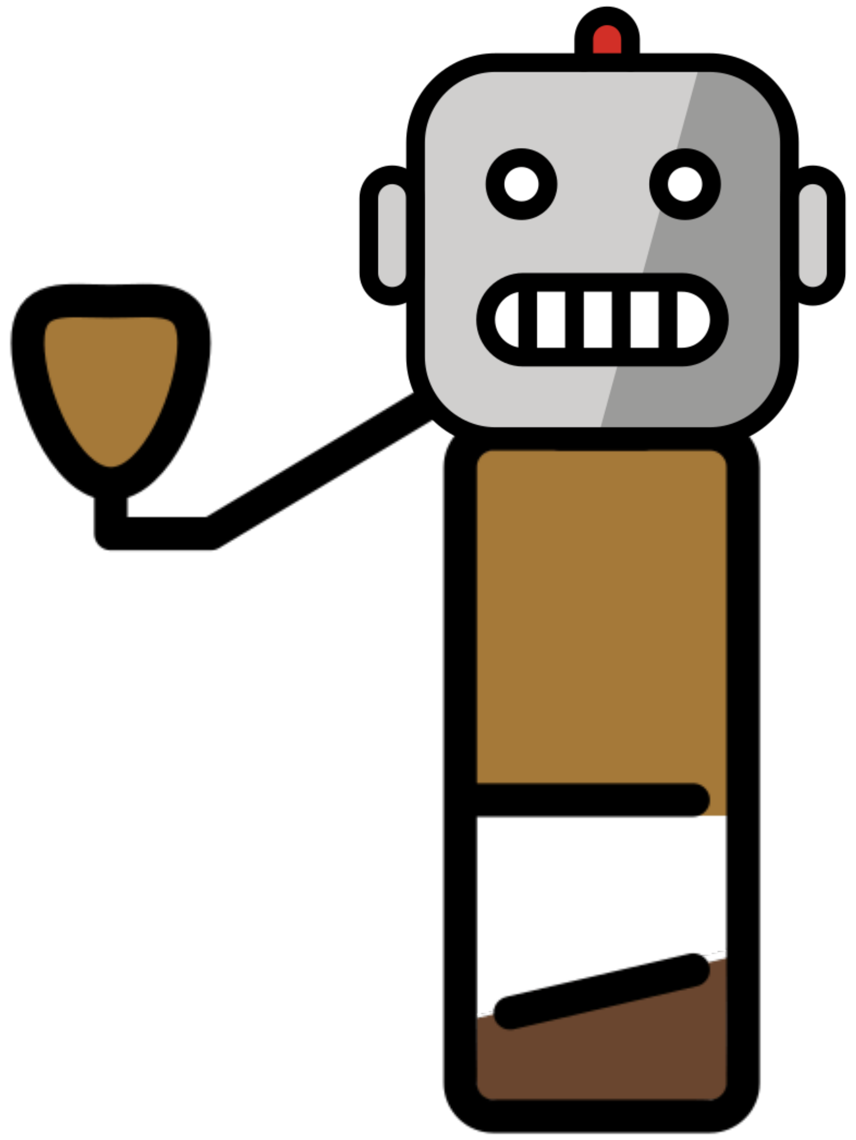

Regression: 2.1m
ViT-Huge: 632m

Taking a more structured approach

Reducing the parameter space

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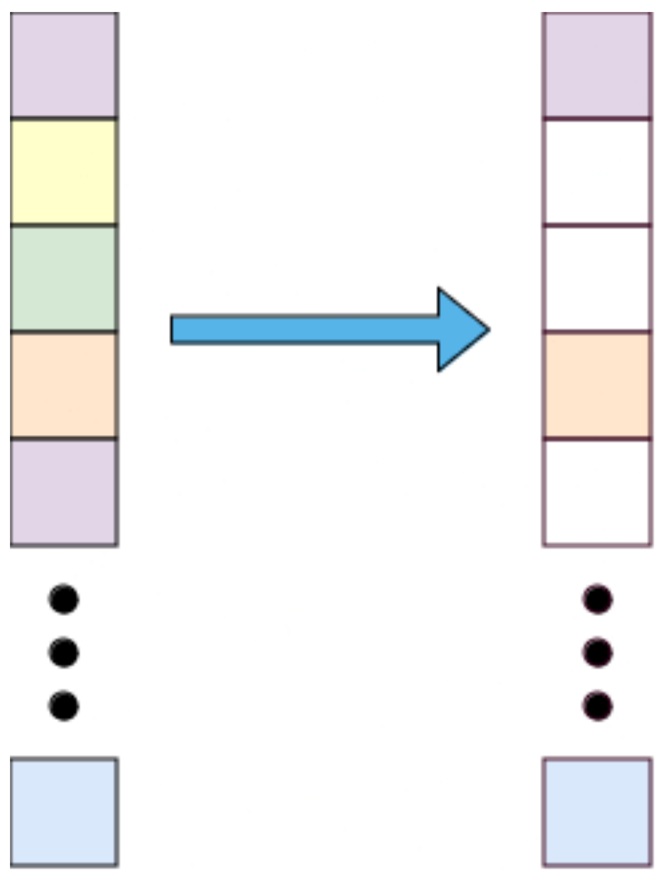

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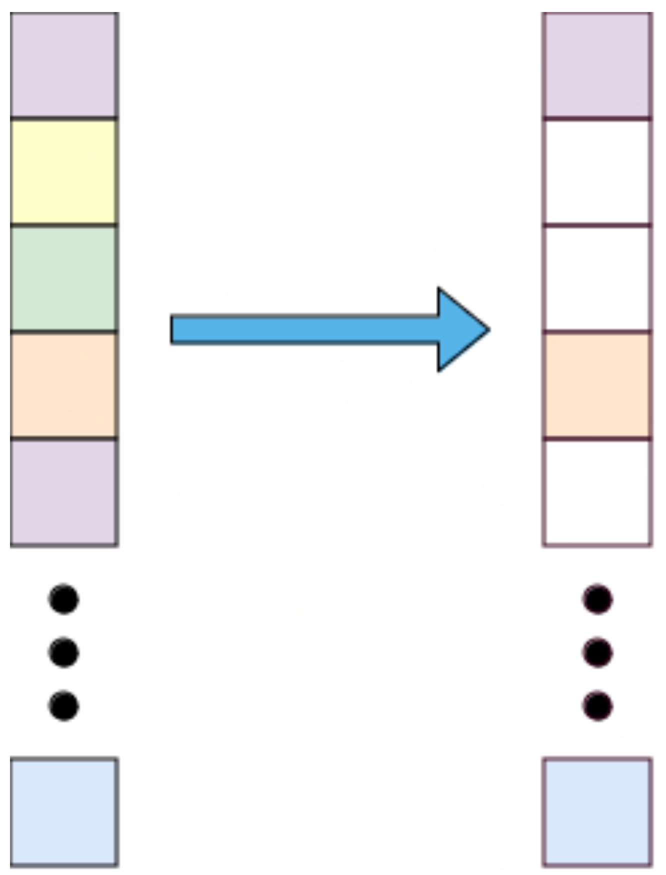

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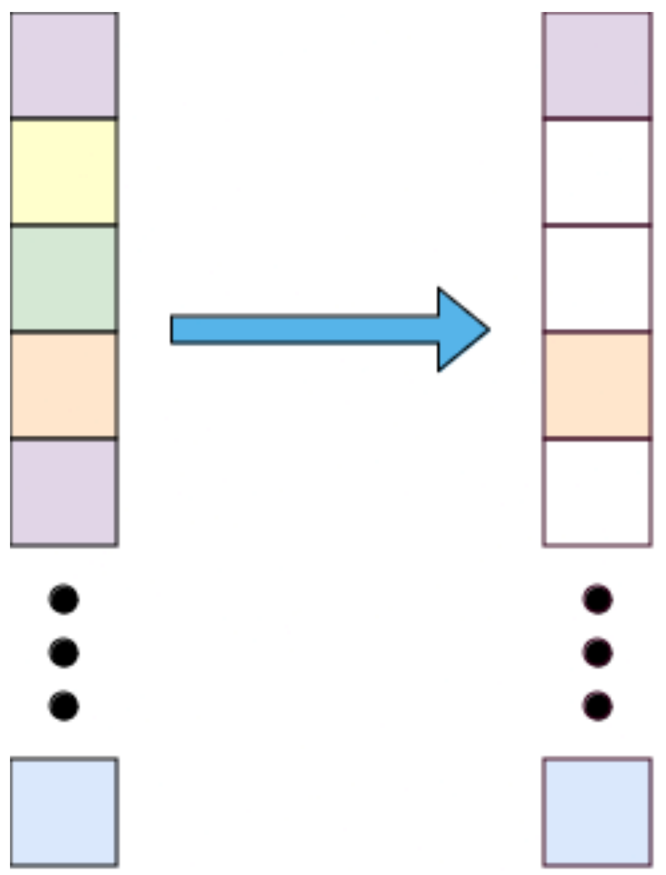

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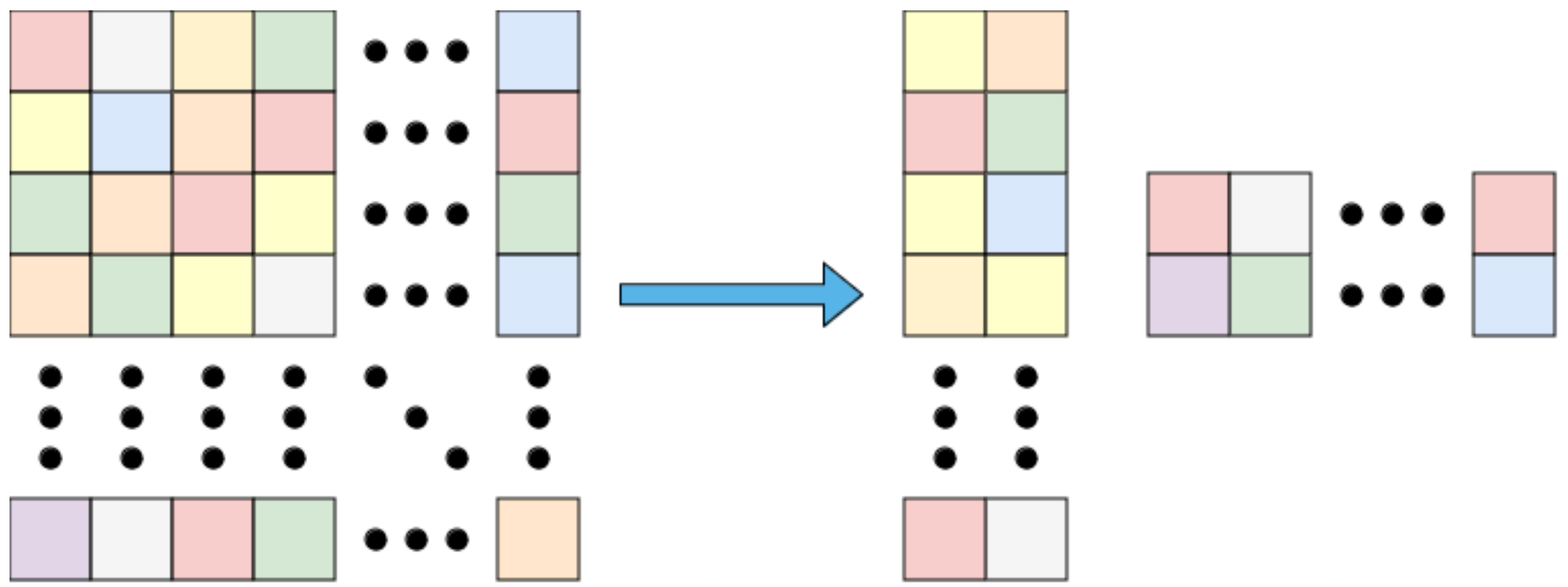
- **Tensors:** a lot more choices!



What's in this talk

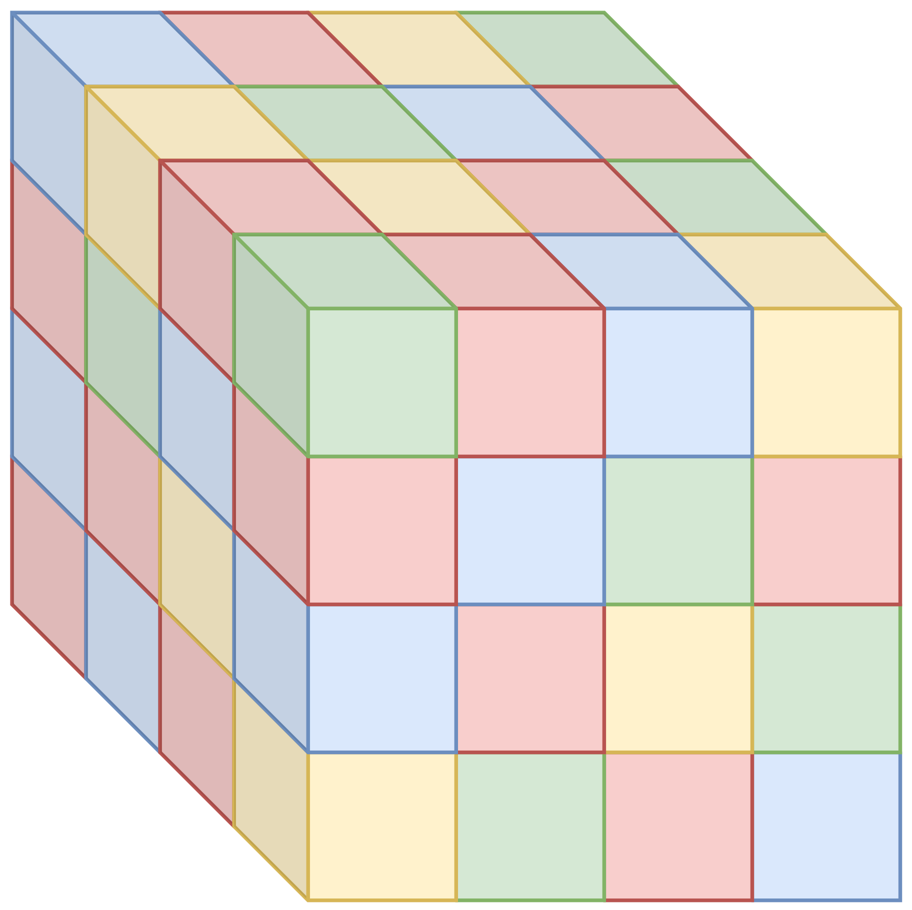
A preview of the rest of the talk

1. Tensor decompositions and where to find them
2. Supervised learning with LSR tensor structures
3. Some current and future directions

Tensor decompositions (old and “new”)

Some tensor terminology

A little jargon is unavoidable…



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Cichocki (2016)
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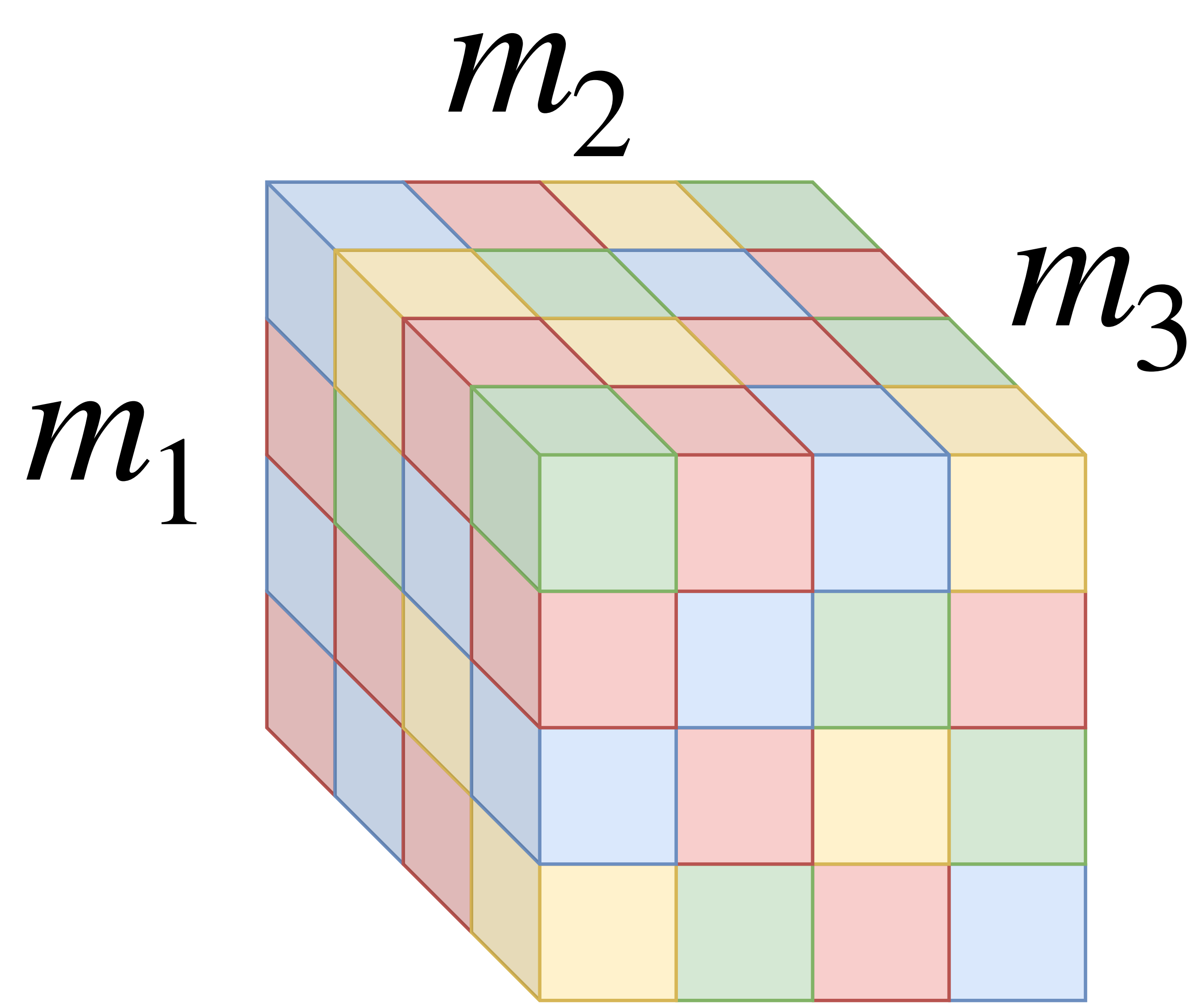

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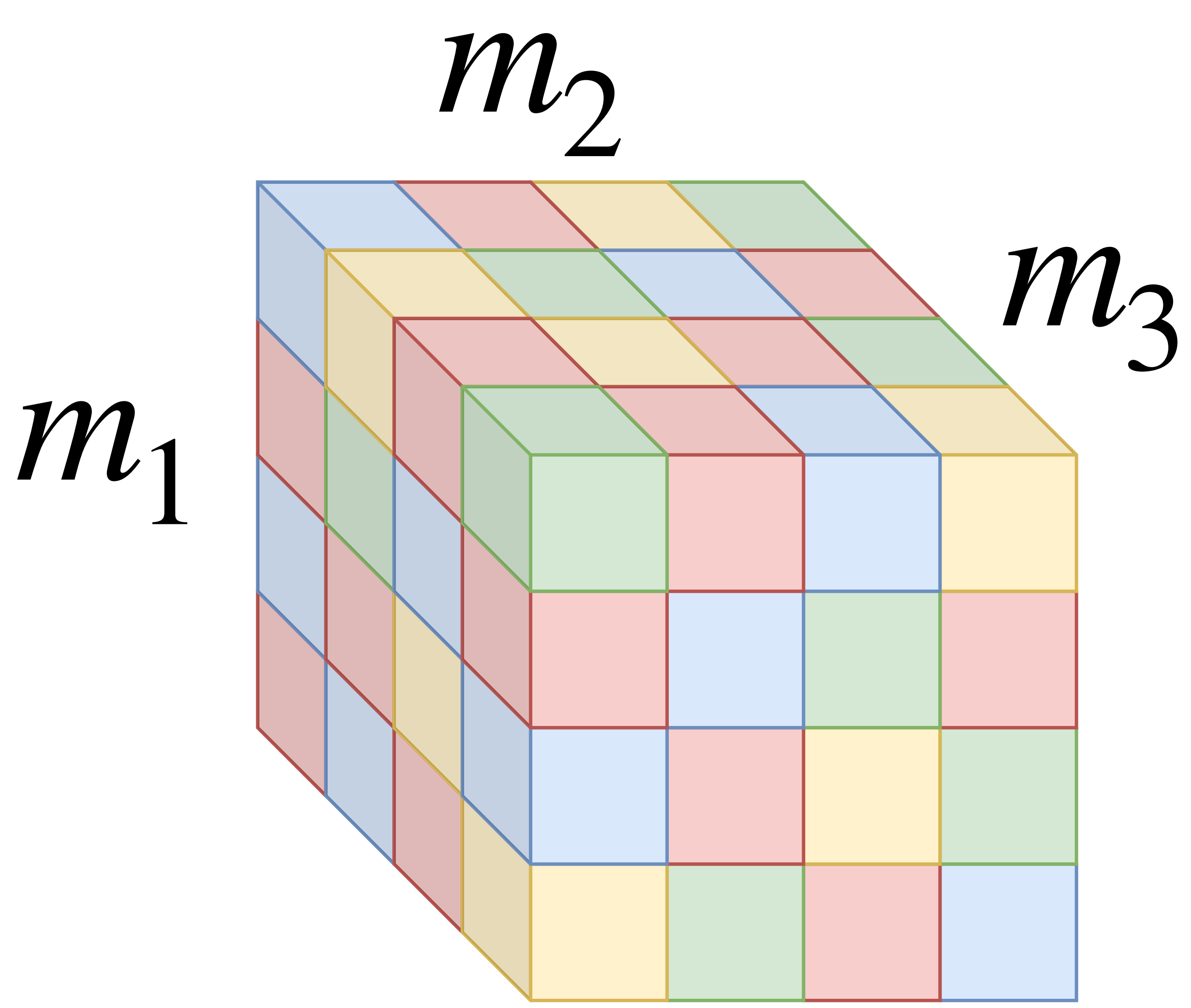

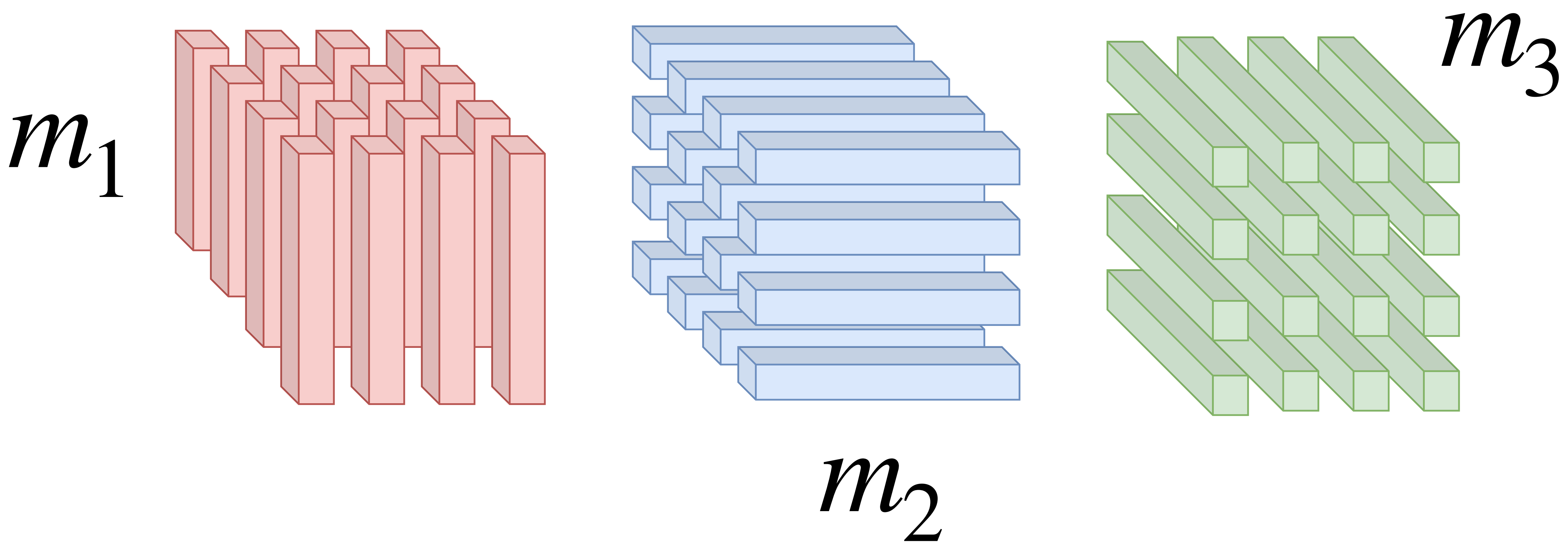

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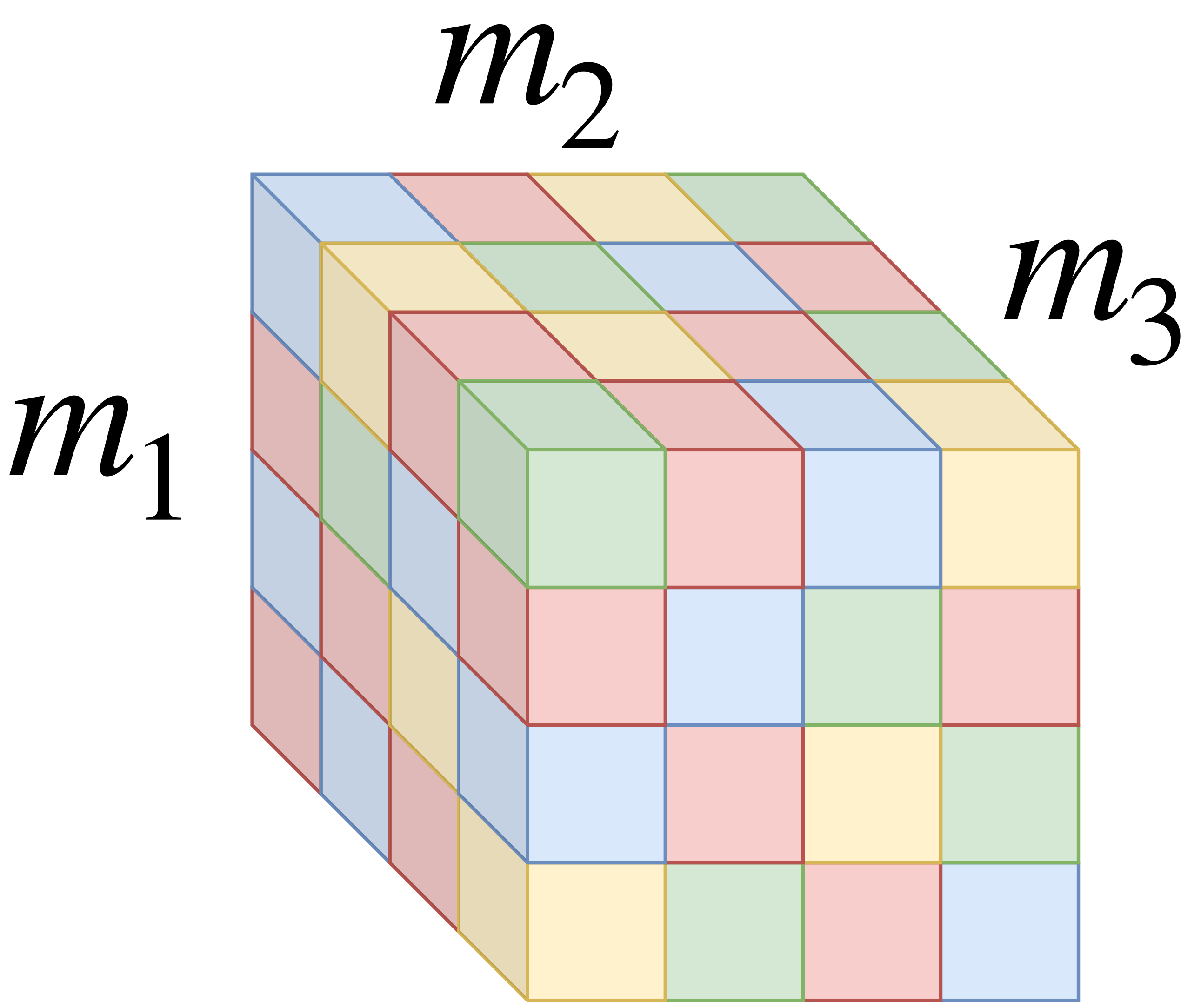
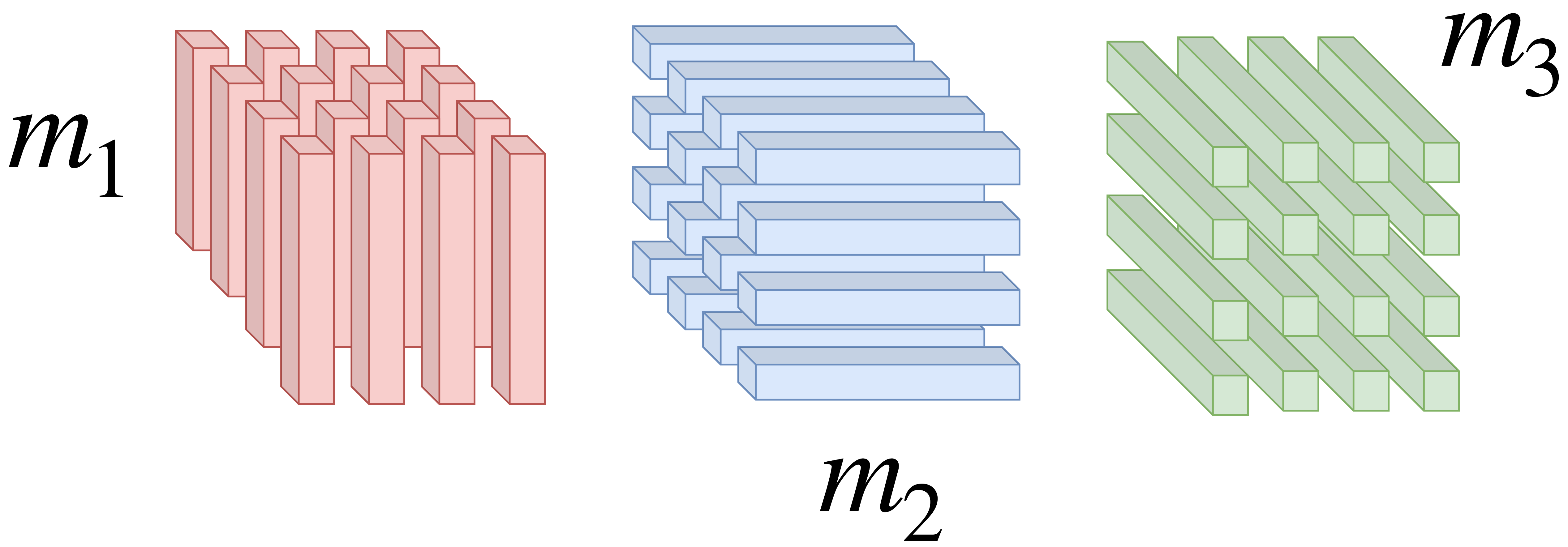


- **Mode:** each coordinate index
- **Order:** the number of modes of the tensor
- **Fibers:** 1-D vectors along each mode



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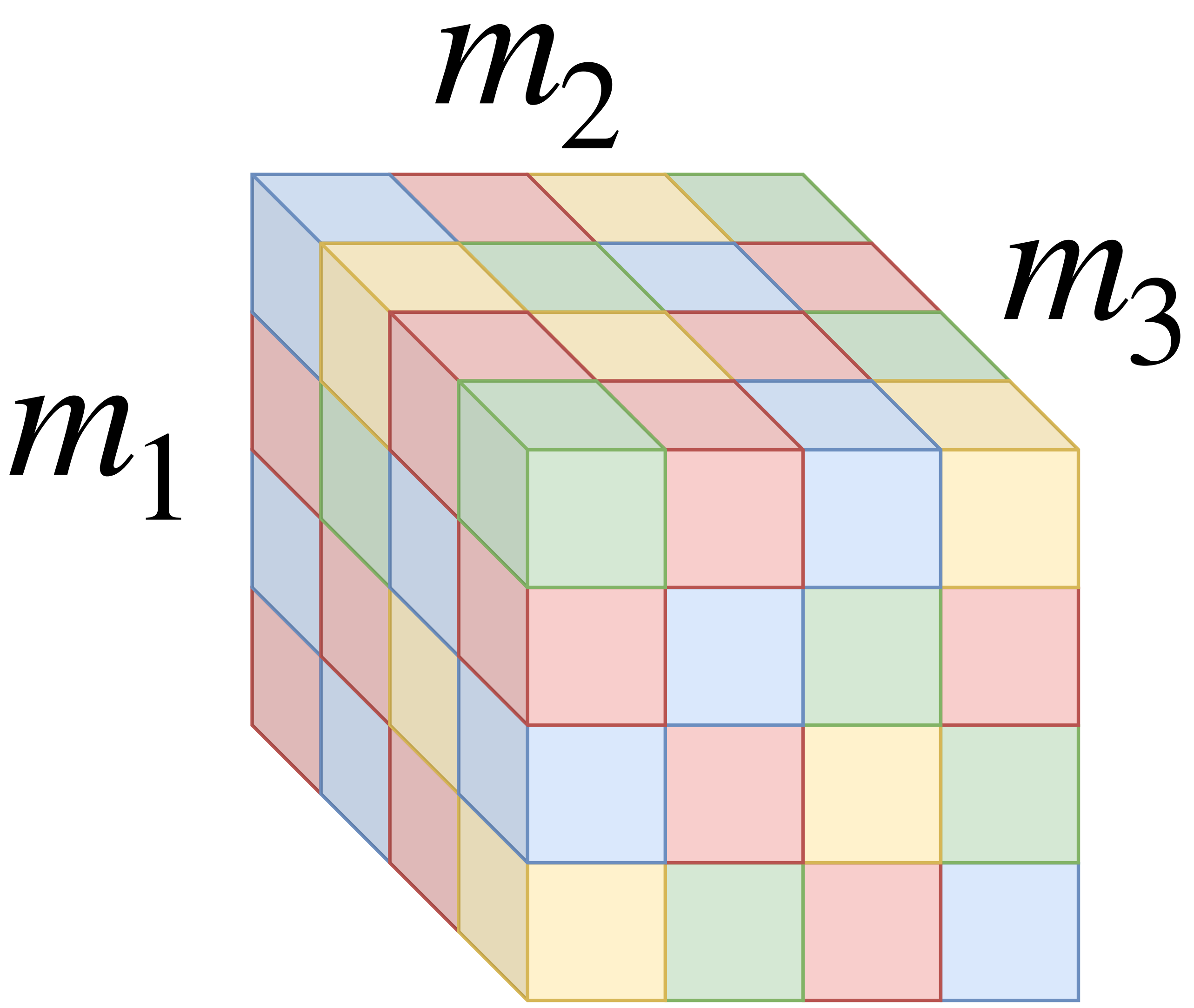

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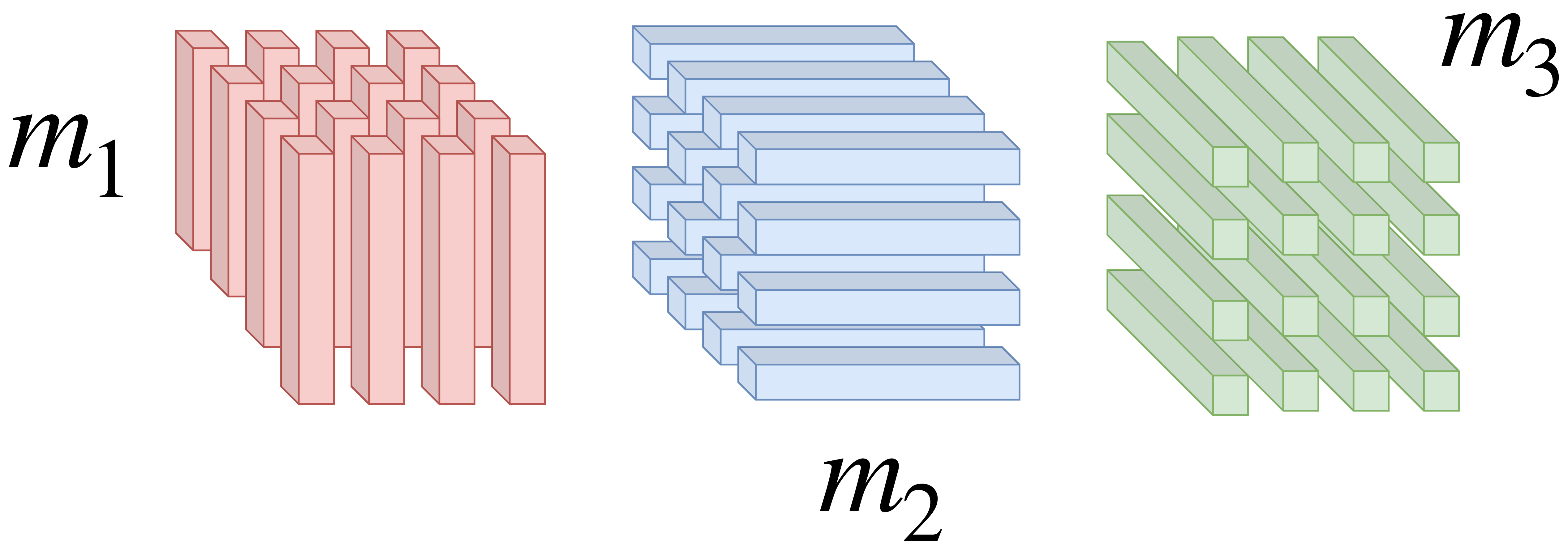

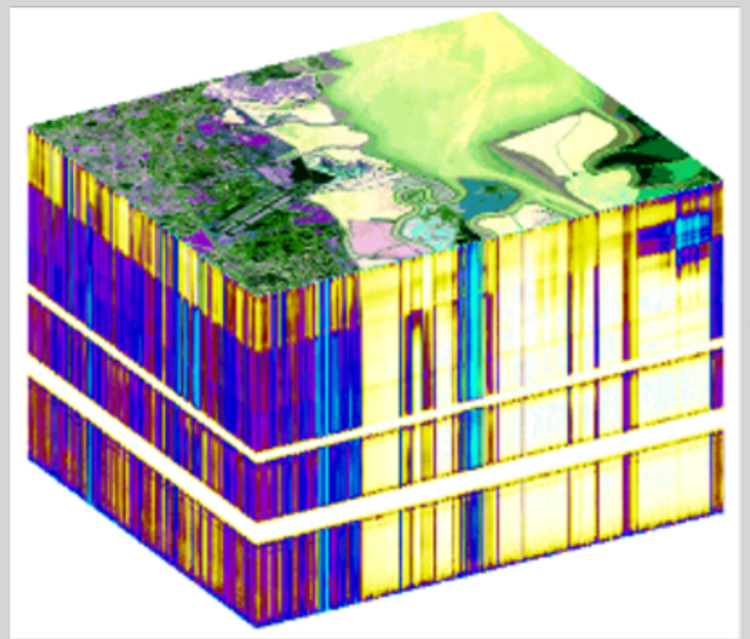

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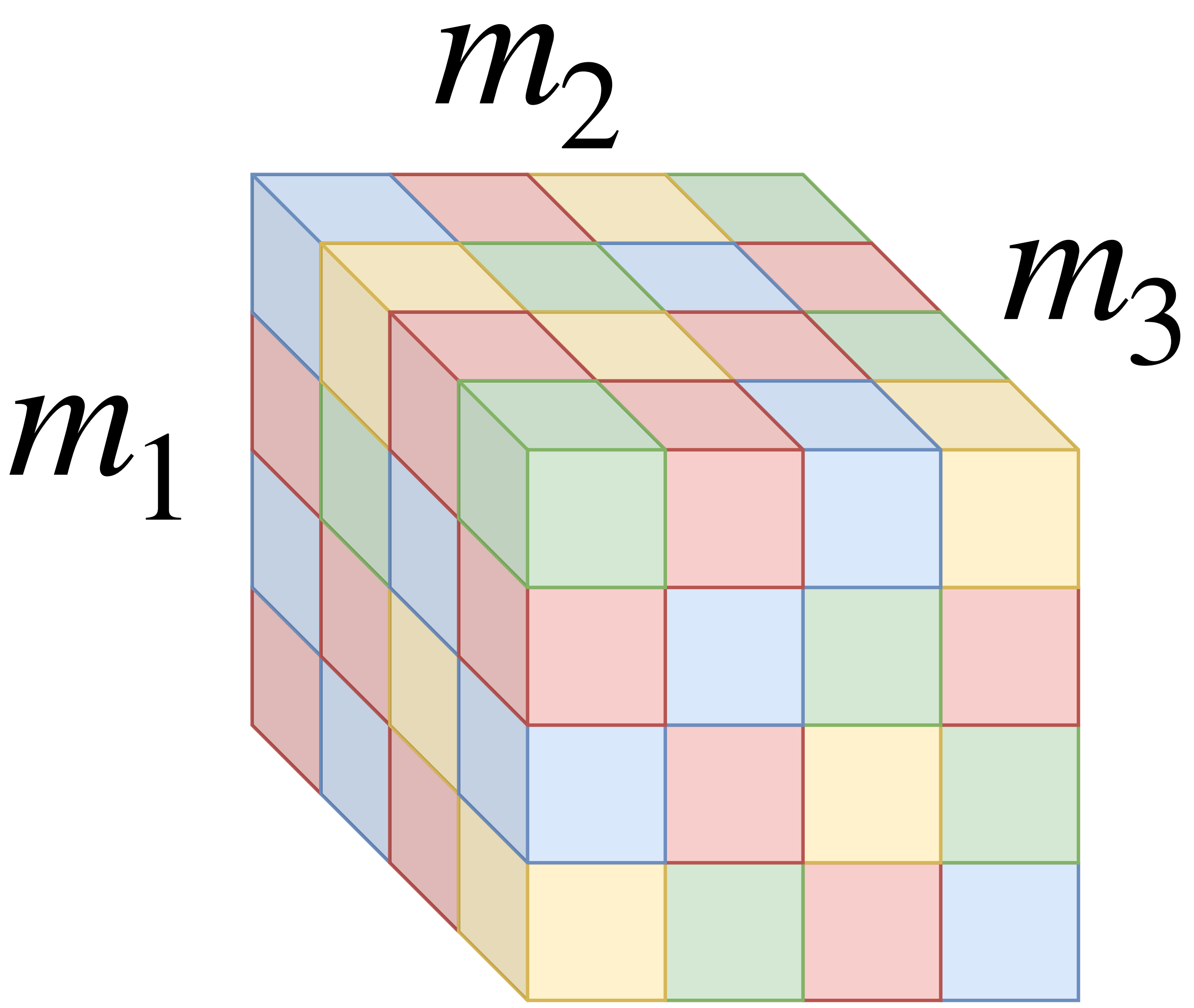

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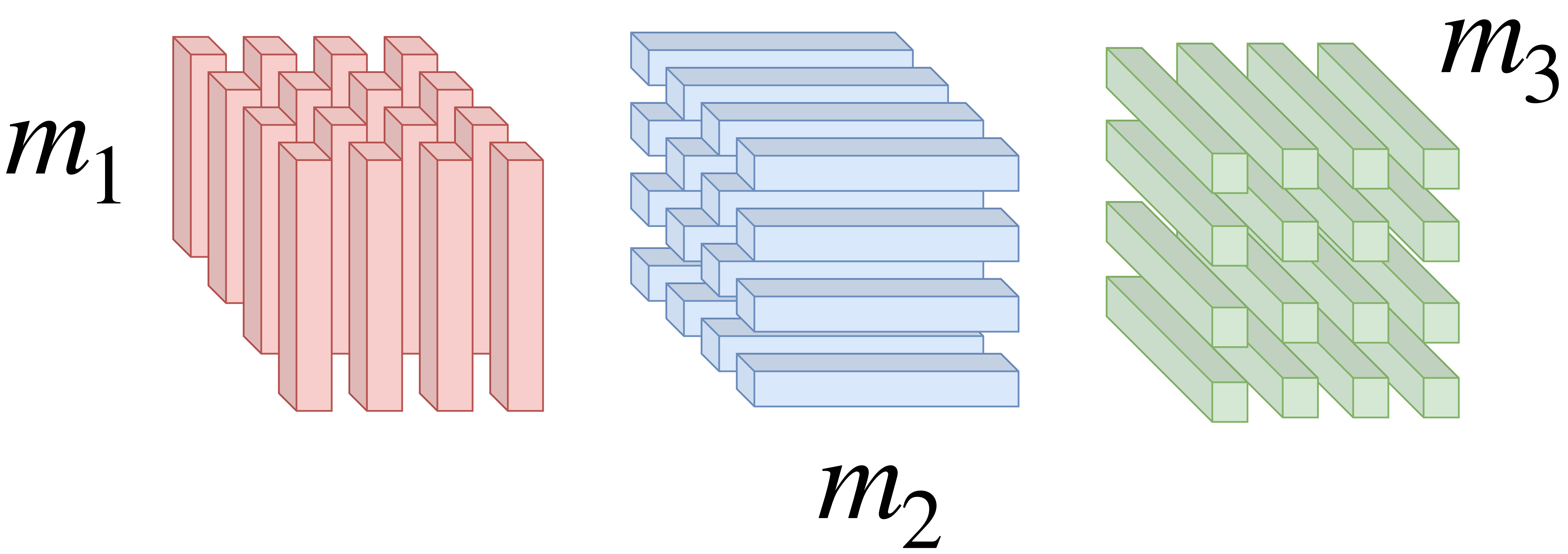

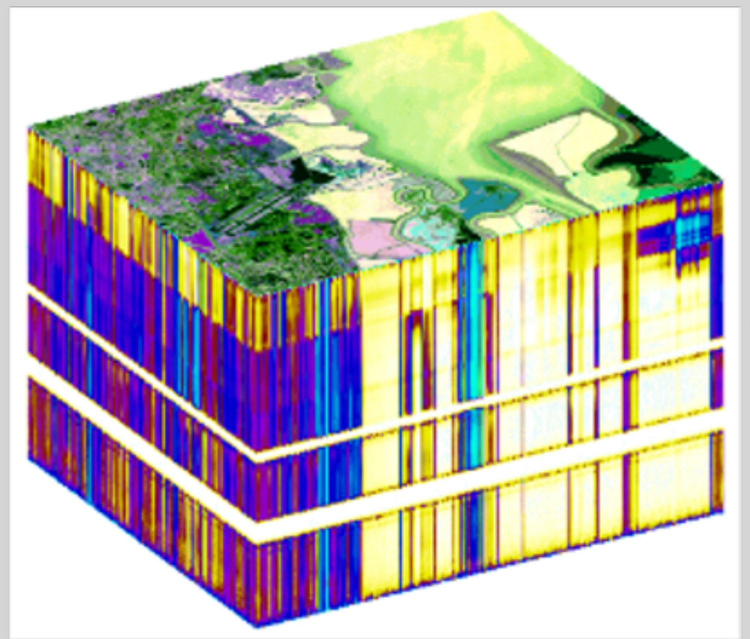

- Mode 1 = spectrum
- Mode 2 = longitude
- Mode 3 = latitude

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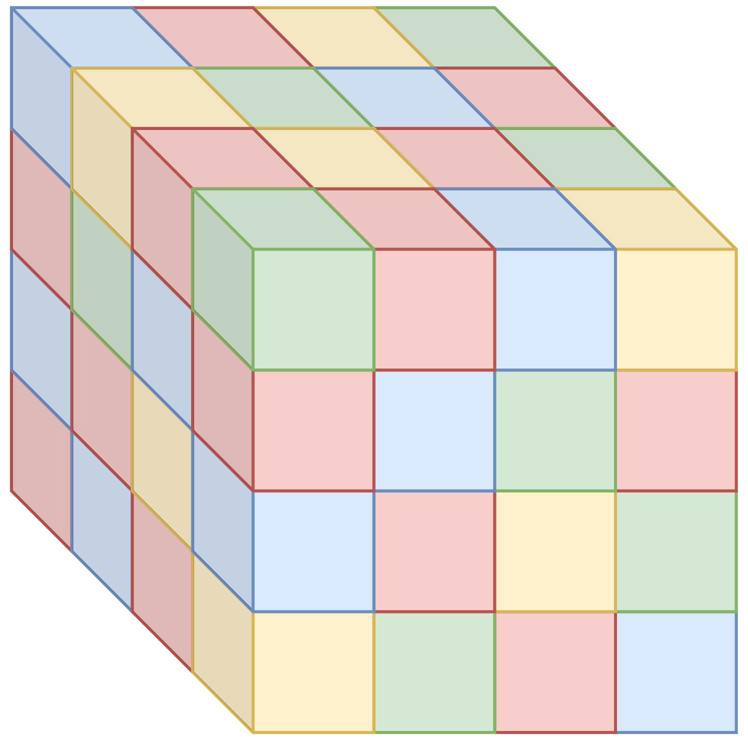
Matrix-tensor products

Mode-wise products



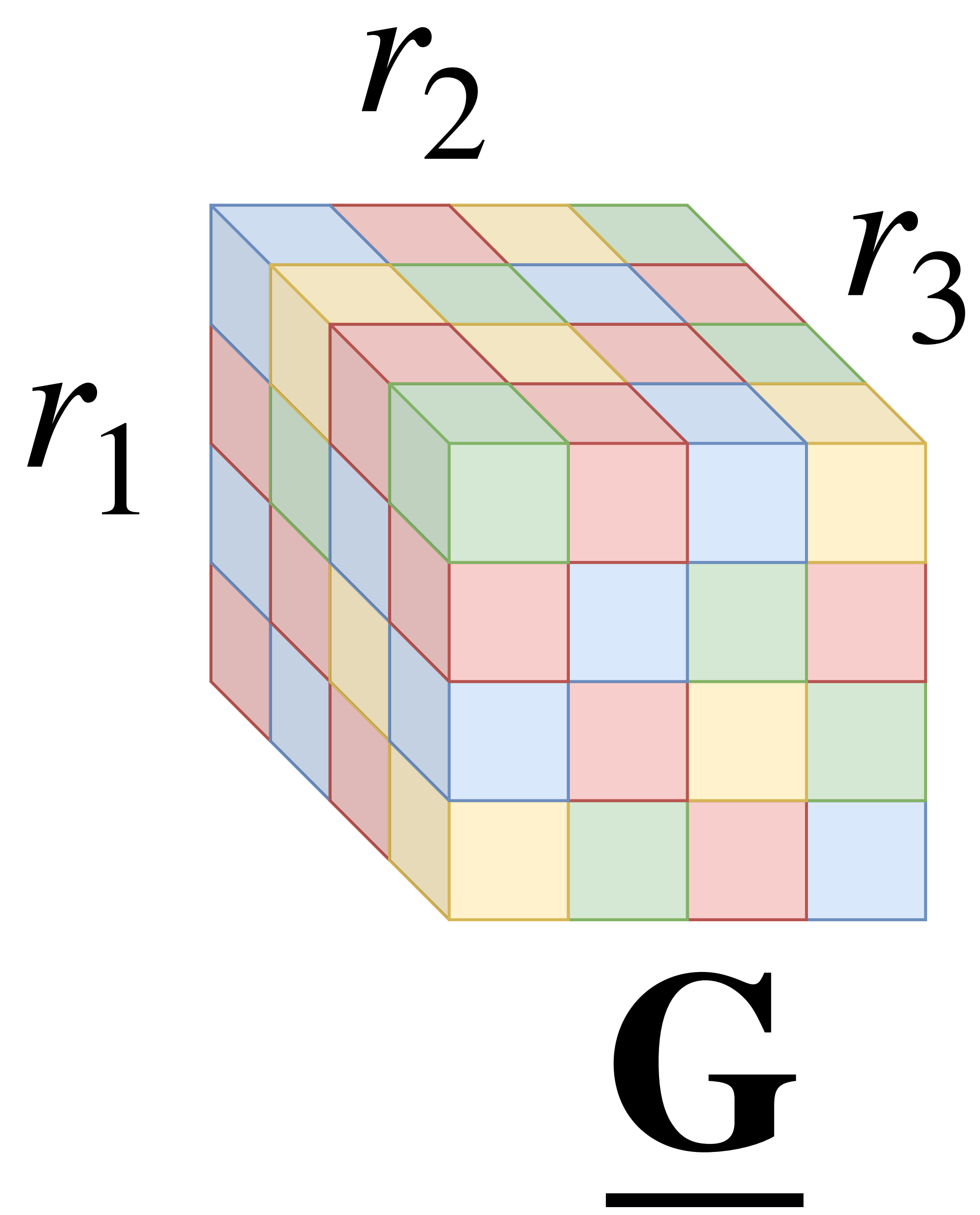
Multiply a tensor $\underline{\mathbf{G}} \in \mathbb{R}^{r_1 \times r_2 \times \cdots \times r_K}$ by a matrix $\mathbf{B}_k \in \mathbb{R}^{m_k \times r_k}$ along mode k:

$$\underline{\mathbf{G}} \times_k \mathbf{B}_k$$

The result is a order-K tensor whose k-th mode is m_k dimensional.

Matrix-tensor products

Mode-wise products



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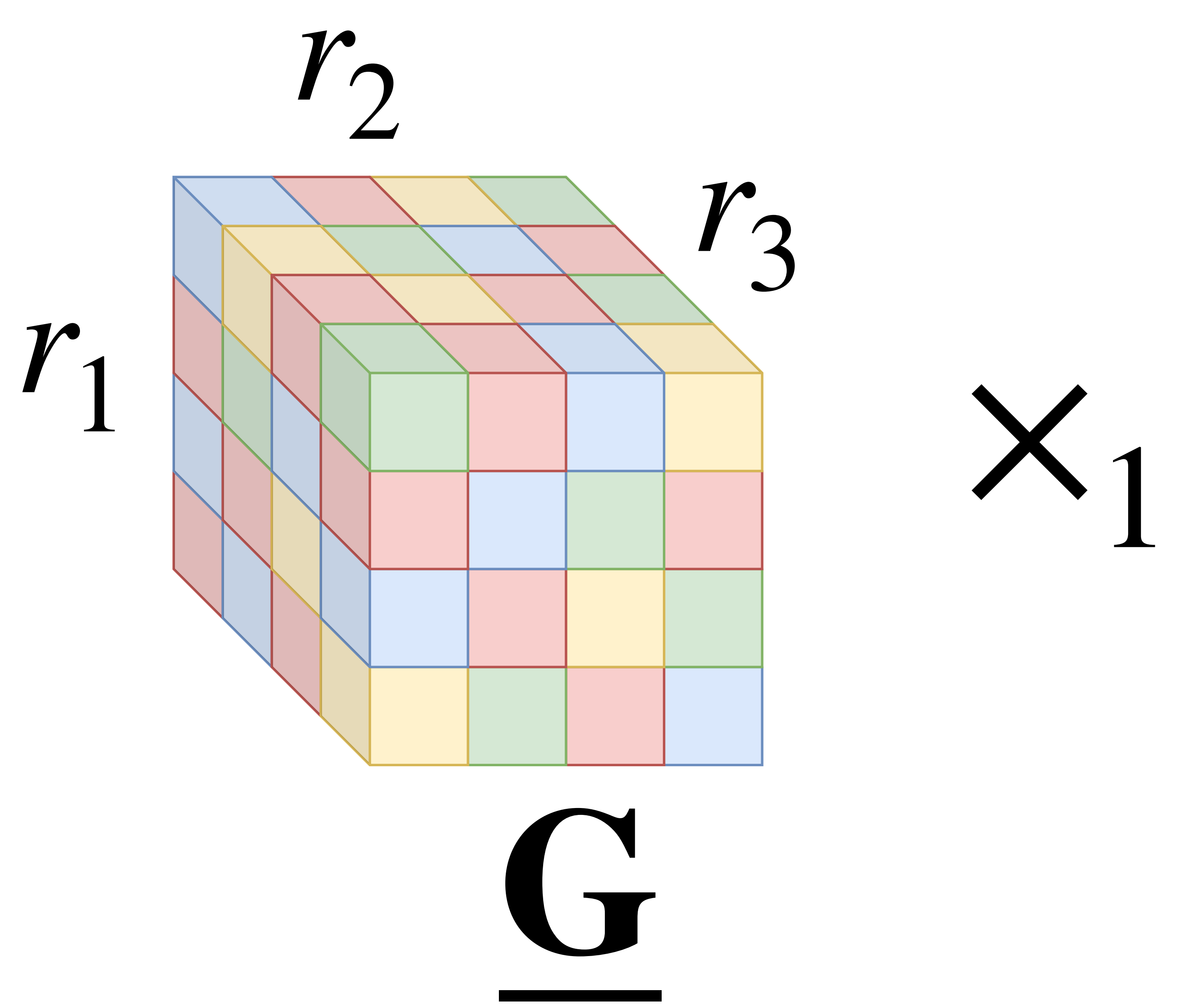
The result is a order-K tensor whose k-th mode is m_k dimensional.

Matrix-tensor products

Mode-wise products



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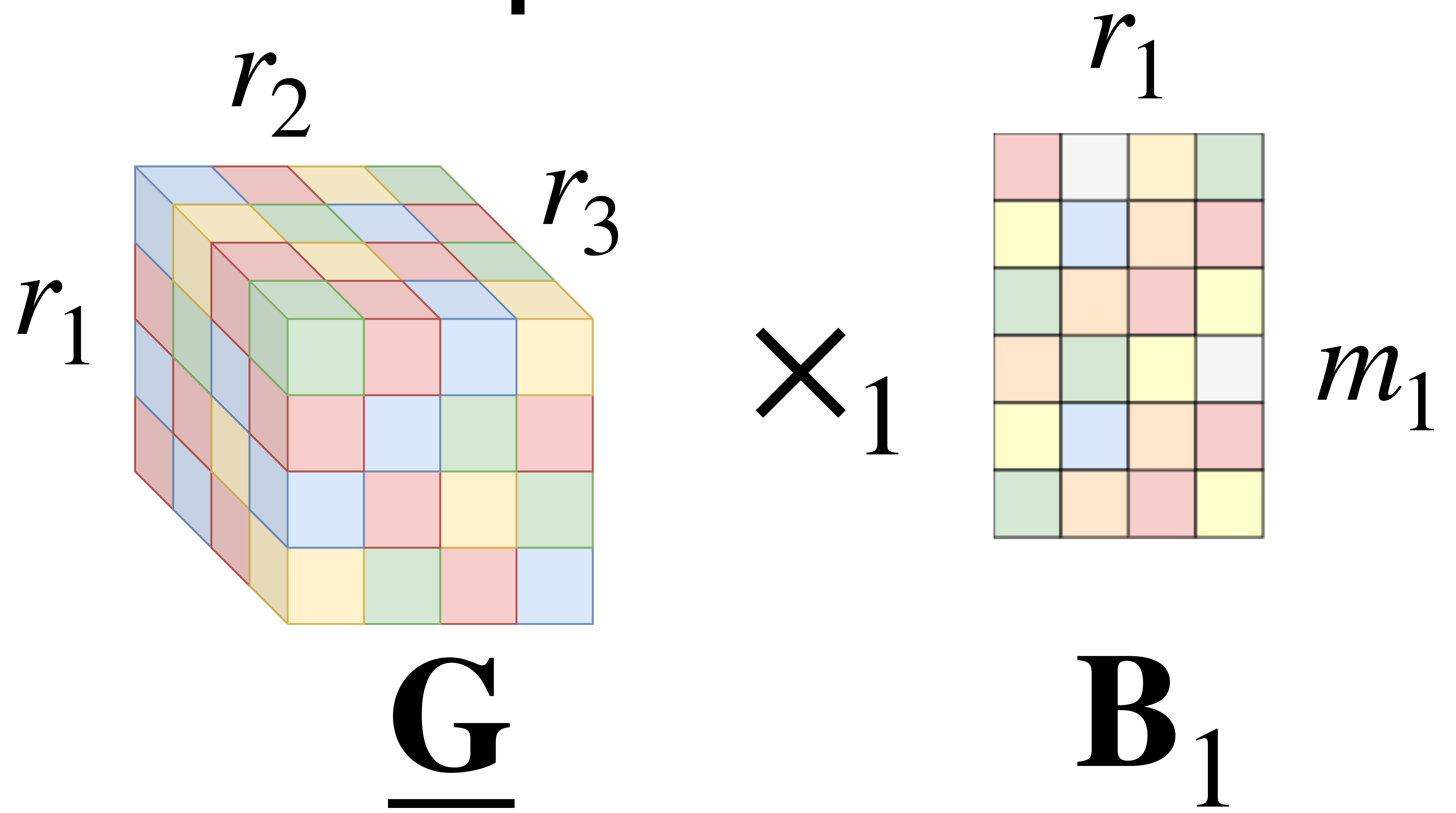

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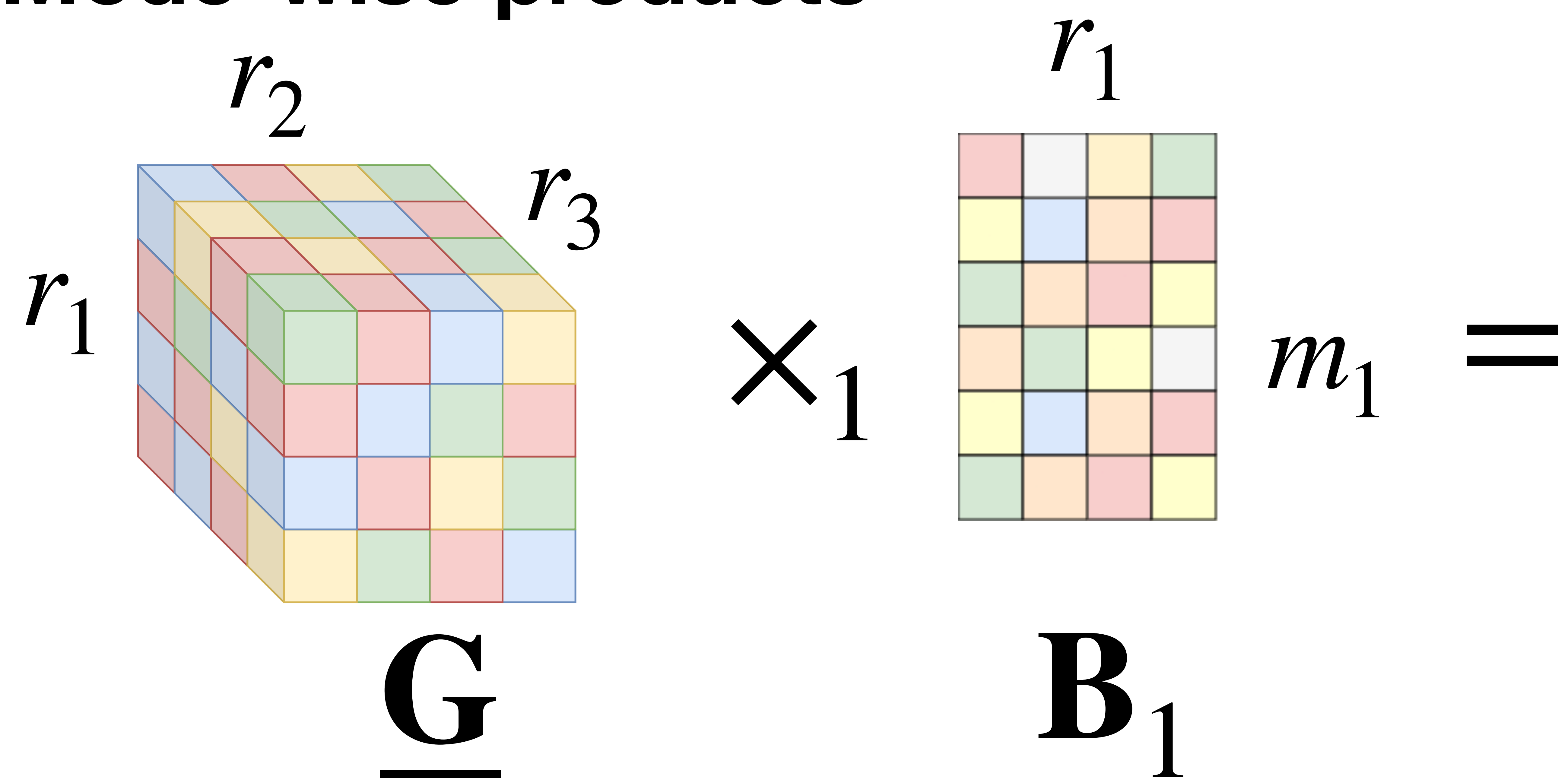

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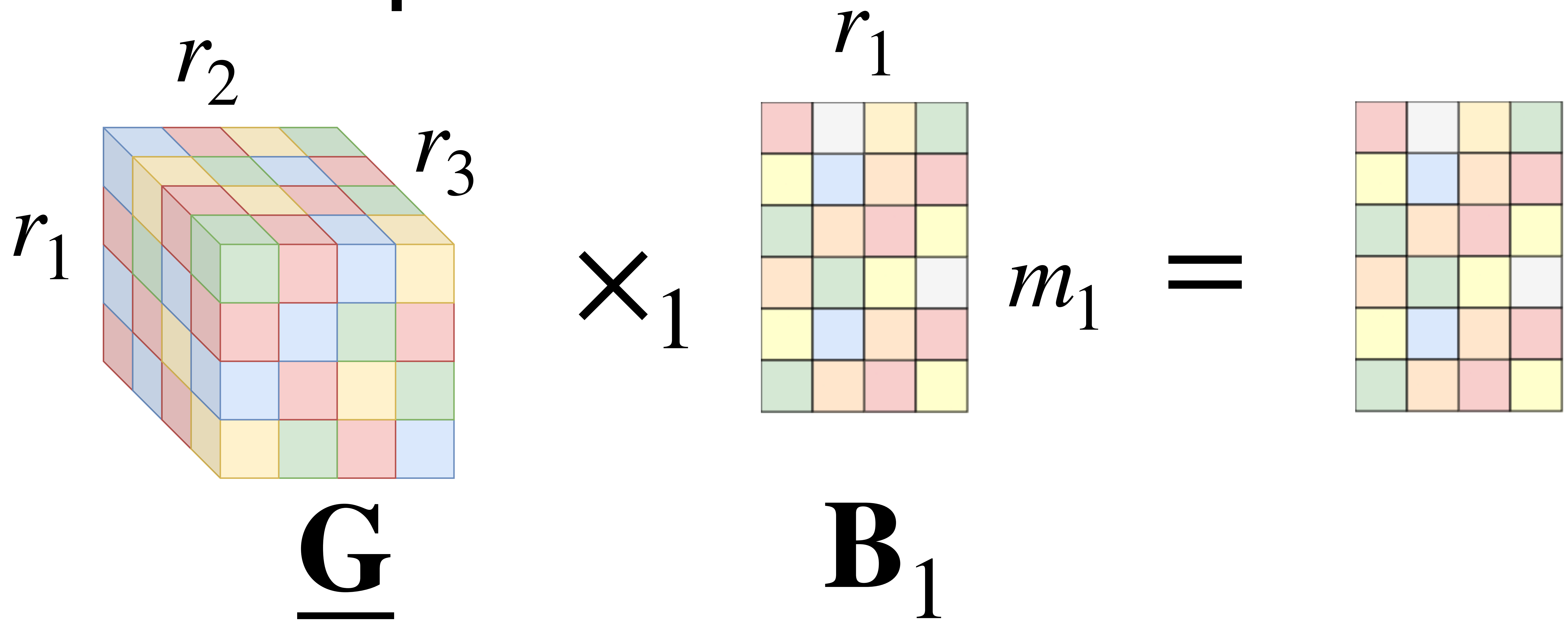

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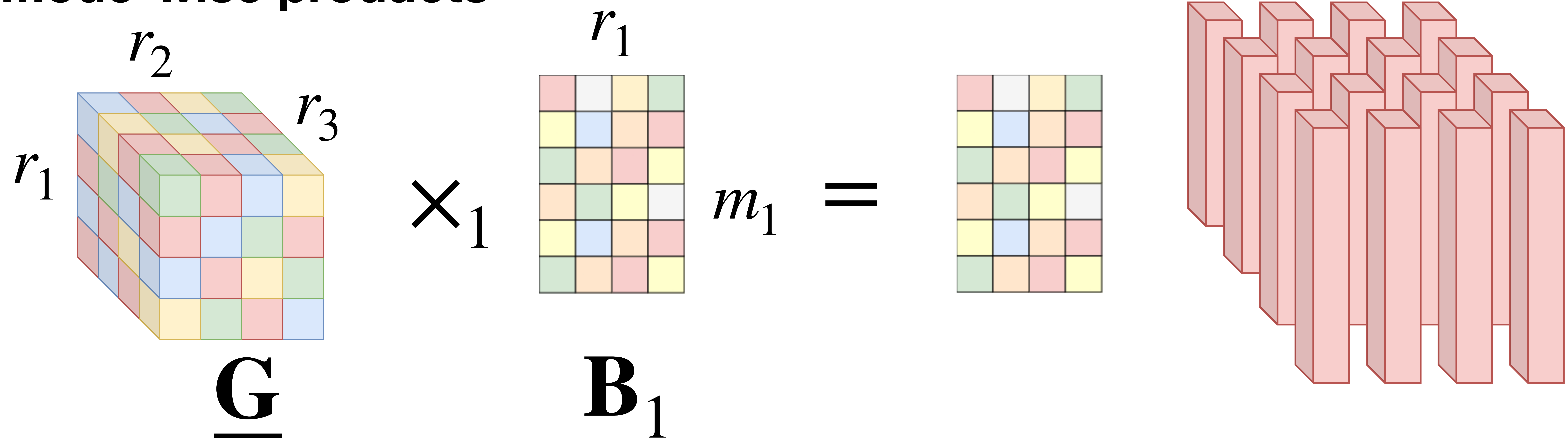

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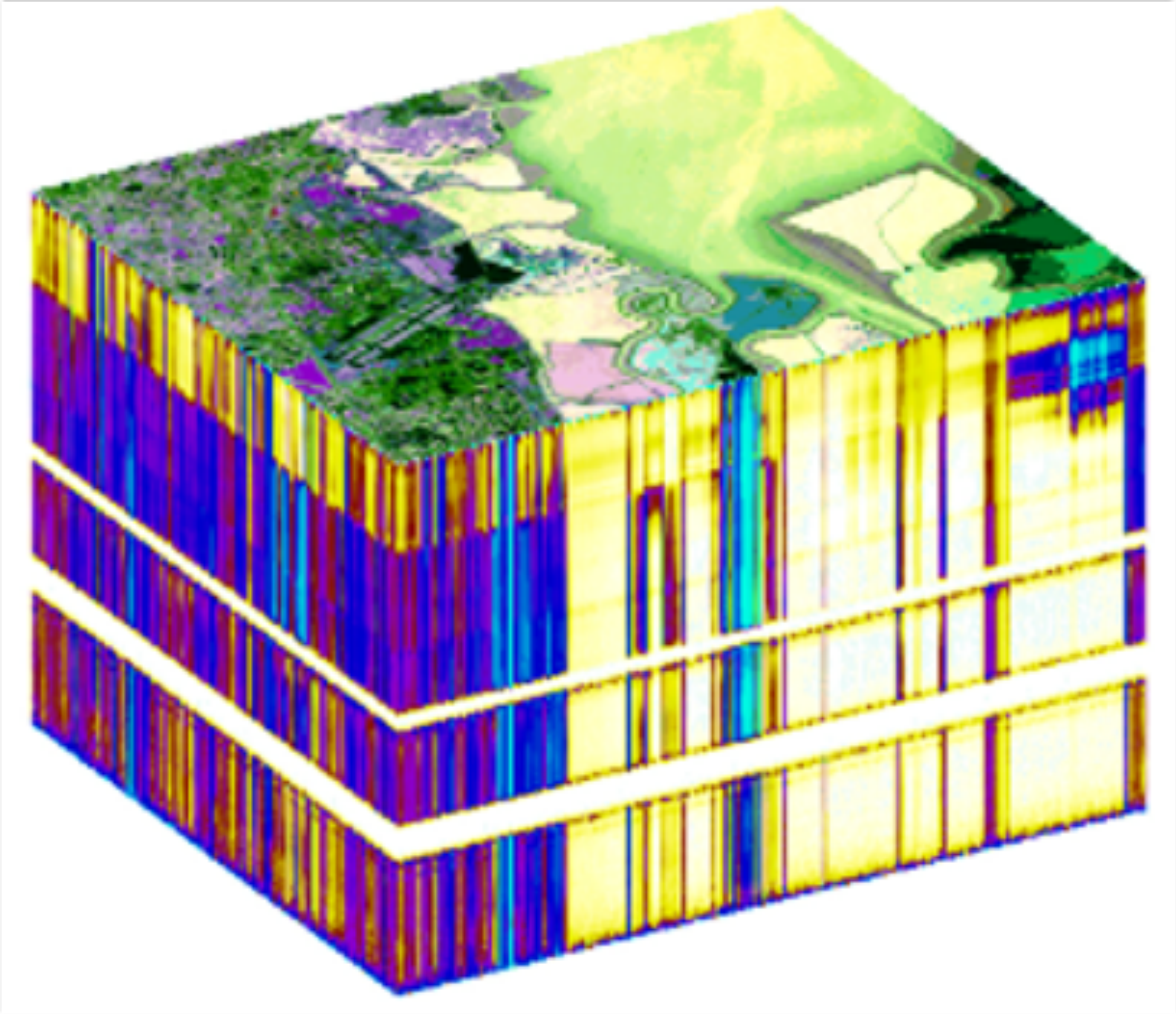
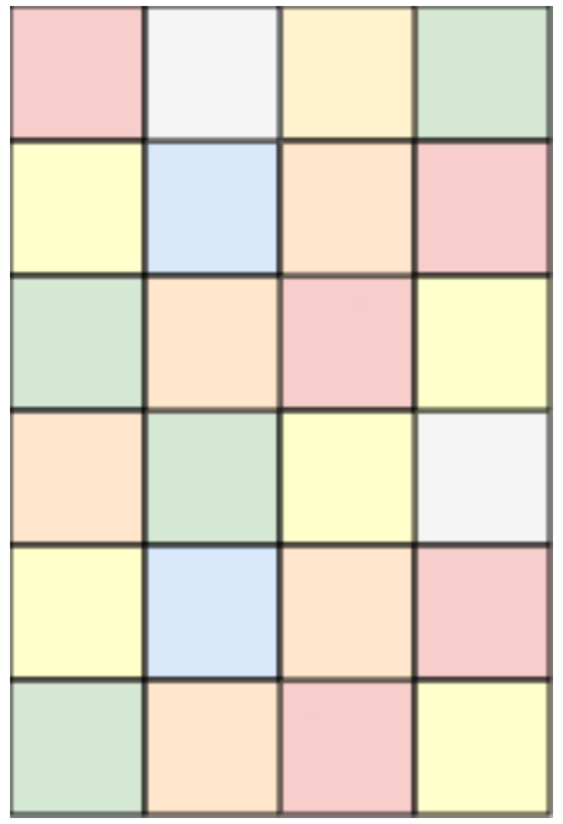
Matrix-tensor product example

Filtering hyperspectral images





$$\underline{\mathbf{X}} \times_1 \mathbf{L}$$

If $\underline{\mathbf{X}}$ is a hyperspectral image and $\mathbf{L}$ is a Discrete Fourier Transform (DFT) matrix corresponding to a lowpass filter, then:

$$\underline{\mathbf{X}} \times_1 \mathbf{L}_1$$

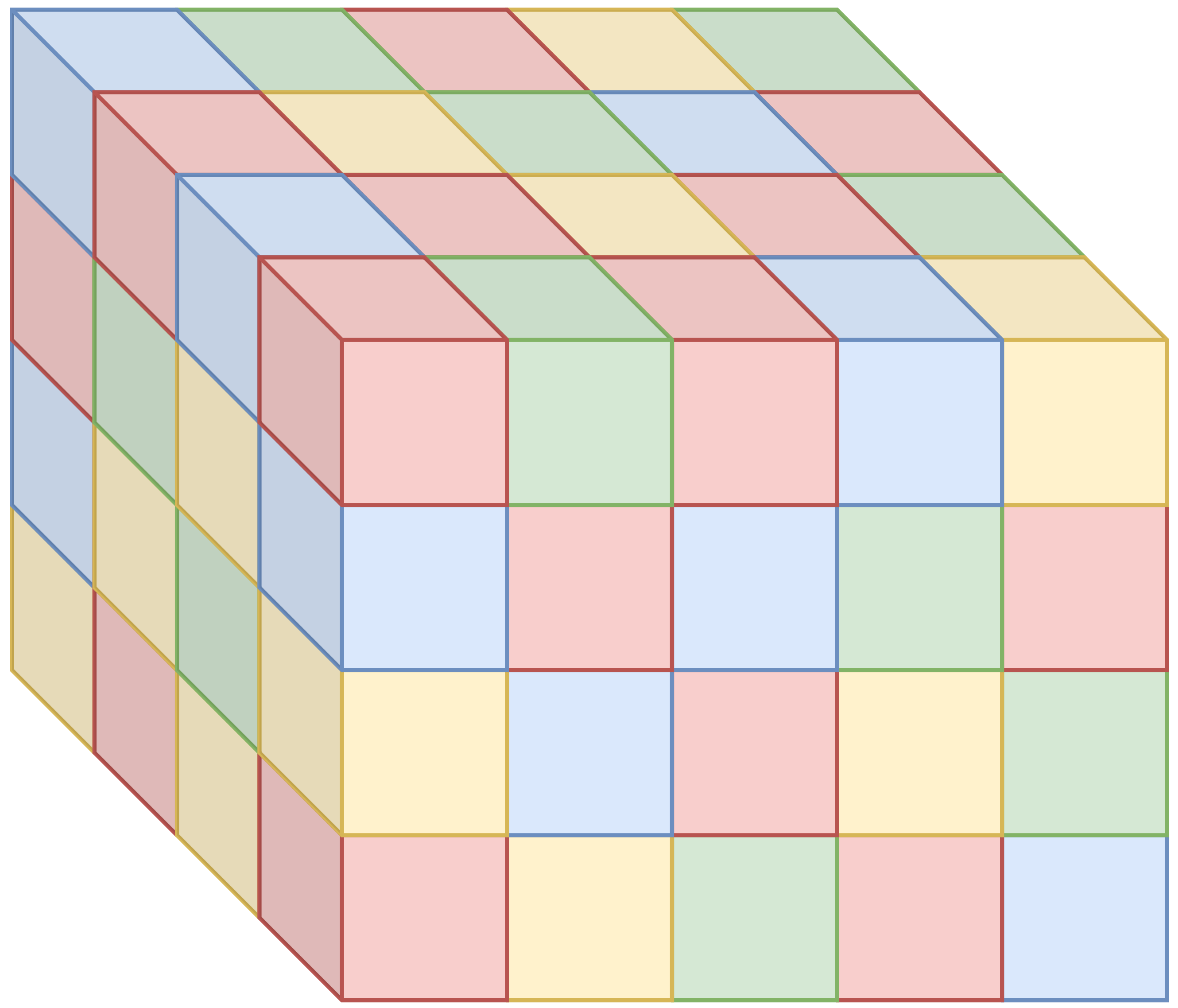
Applies the lowpass filter to the fiber (spectrum) at each physical location in space.

Chaining matrix-tensor products

Processing multiple modes

Chaining matrix-tensor products

Processing multiple modes



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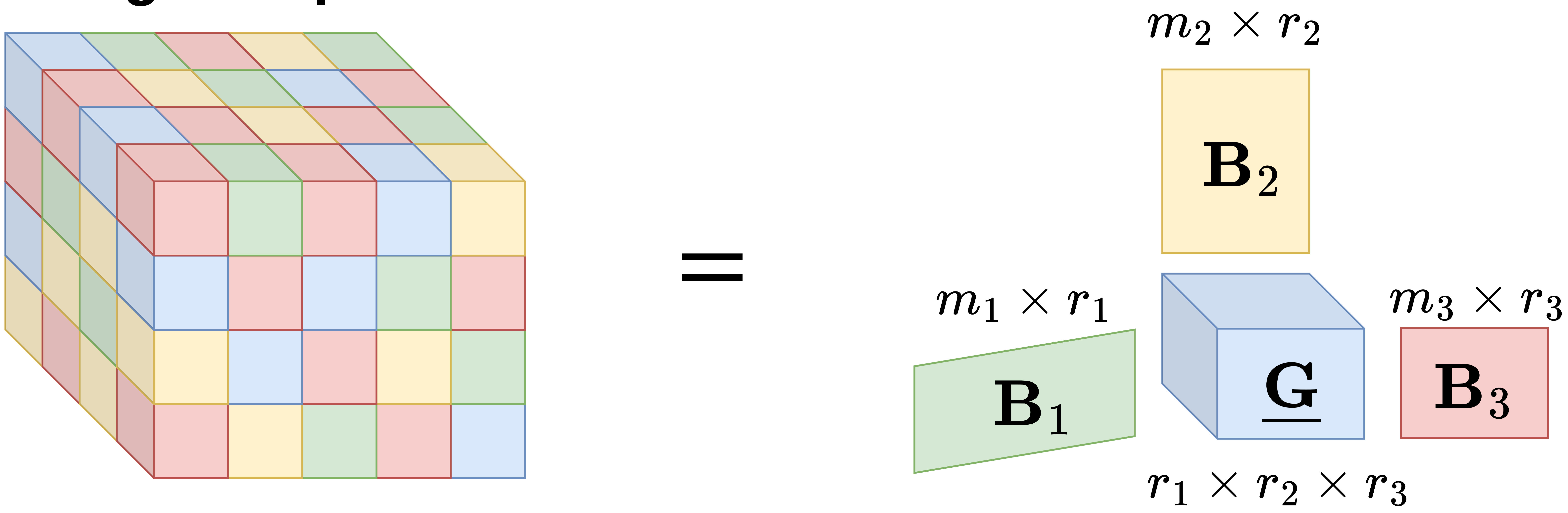

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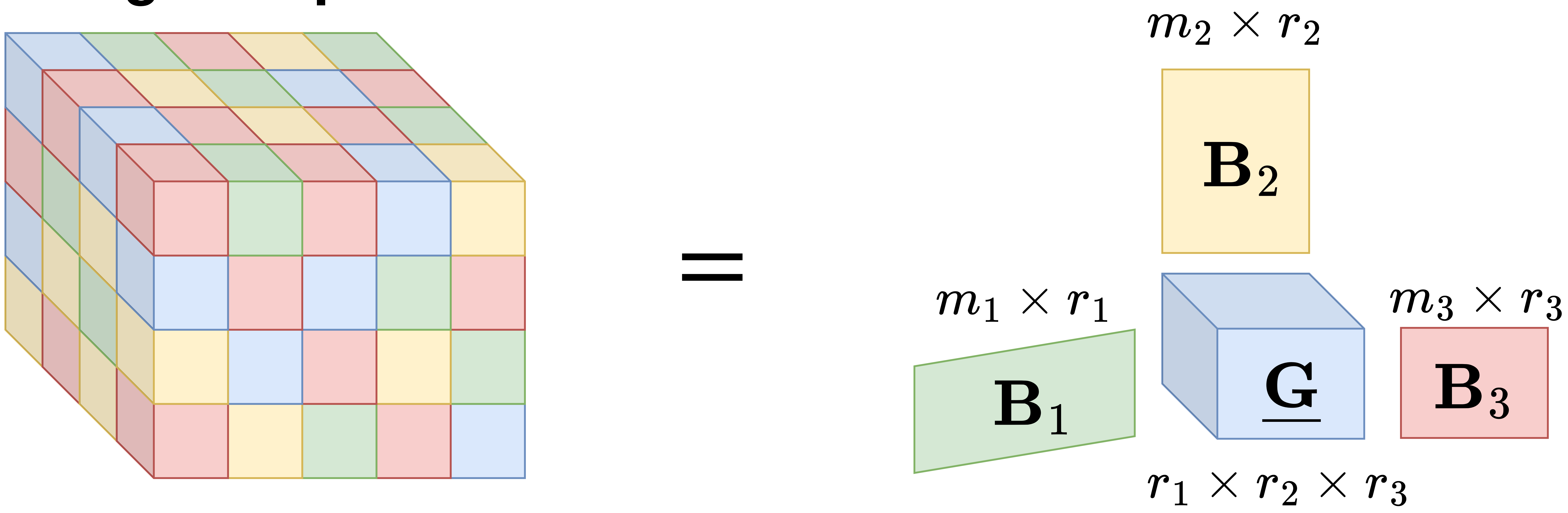


We can change the shape of a tensor with repeated matrix-tensor products

$$\underline{\mathbf{G}} \times_1 \mathbf{B}_1 \times_2 \mathbf{B}_2 \cdots \times_K \mathbf{B}_K = \underline{\mathbf{X}} \in \mathbb{R}^{m_1 \times m_2 \cdots \times m_K}$$

Tensor Rank(s) and Tensor Decompositions/Factorizations

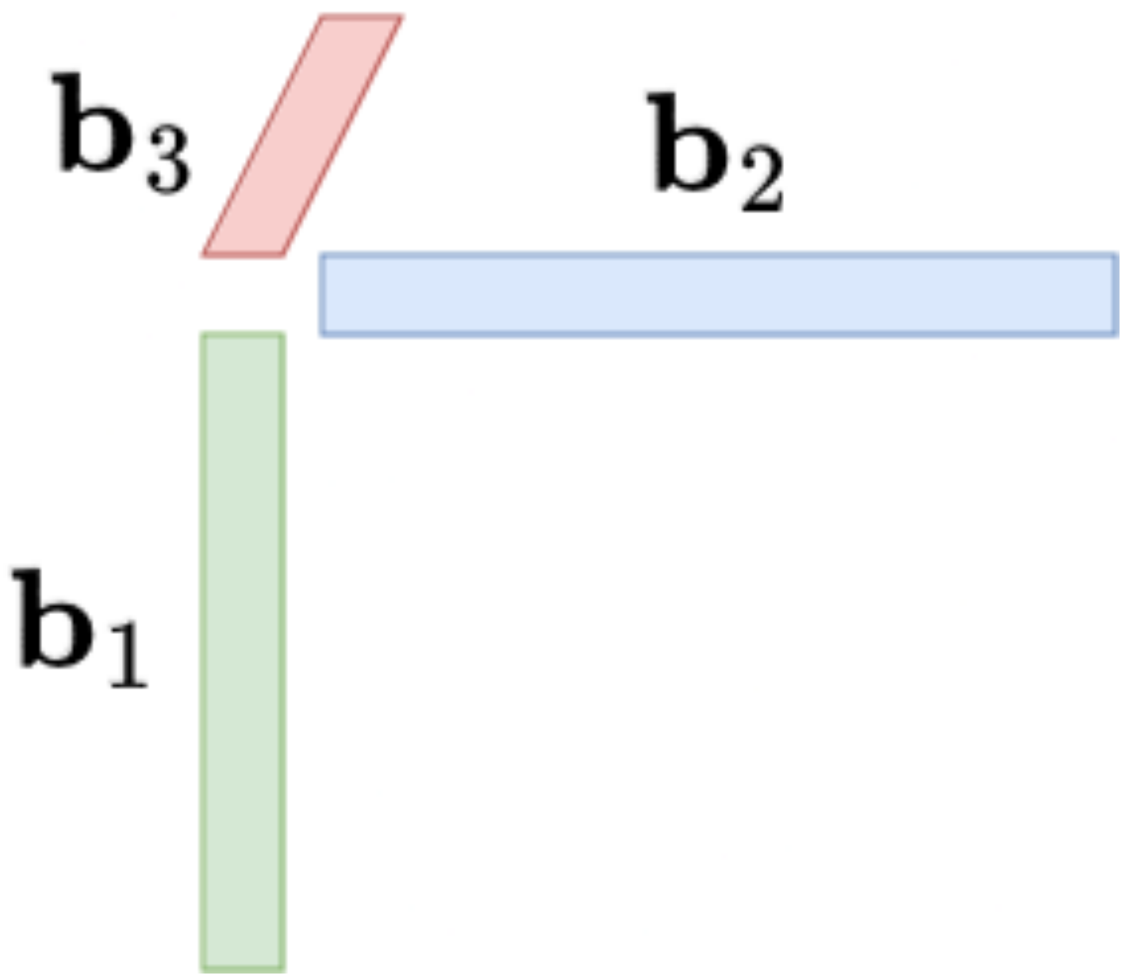
Rank-1 tensors are outer products

Trying to get a handle on rank

Rank-1 tensors are outer products

Trying to get a handle on rank

- 2D: a rank-1 *matrix*



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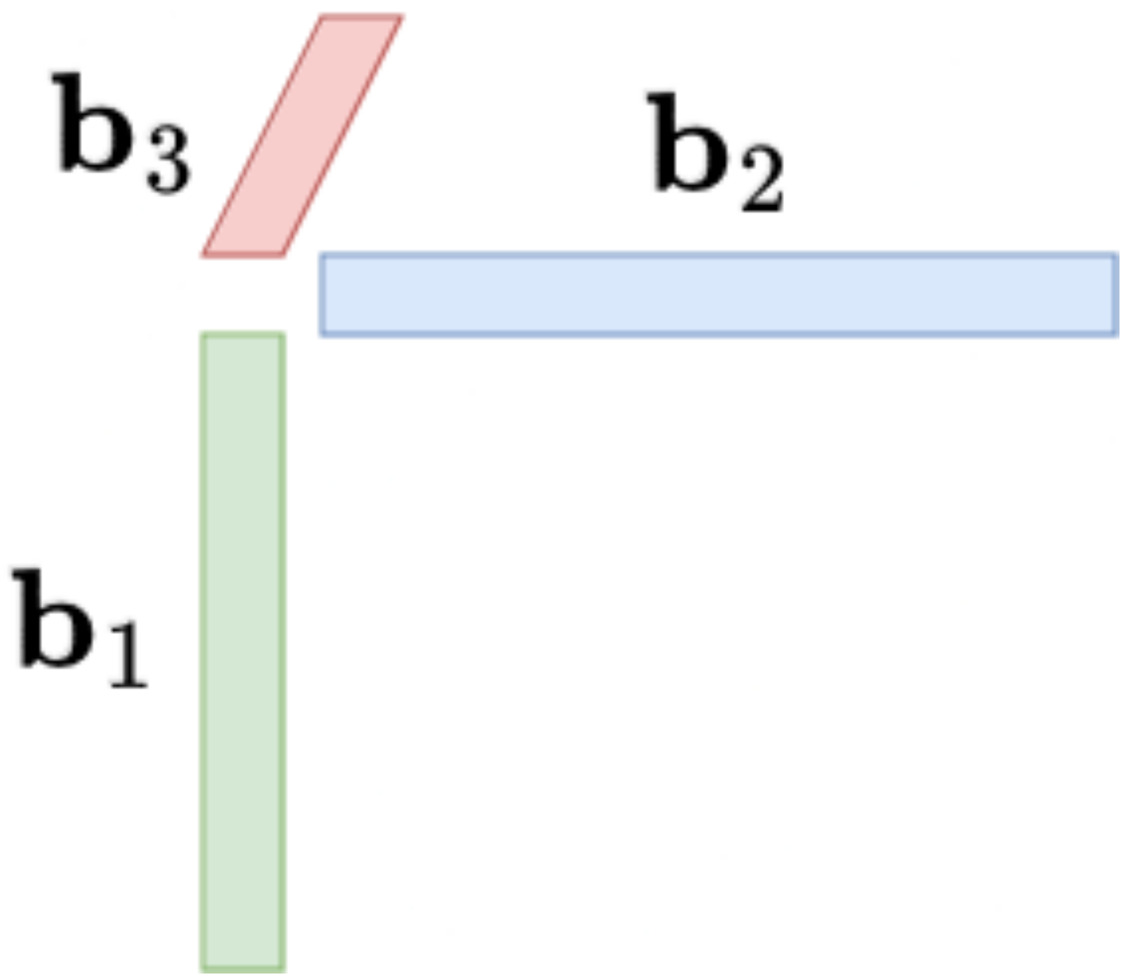
Trying to get a handle on rank

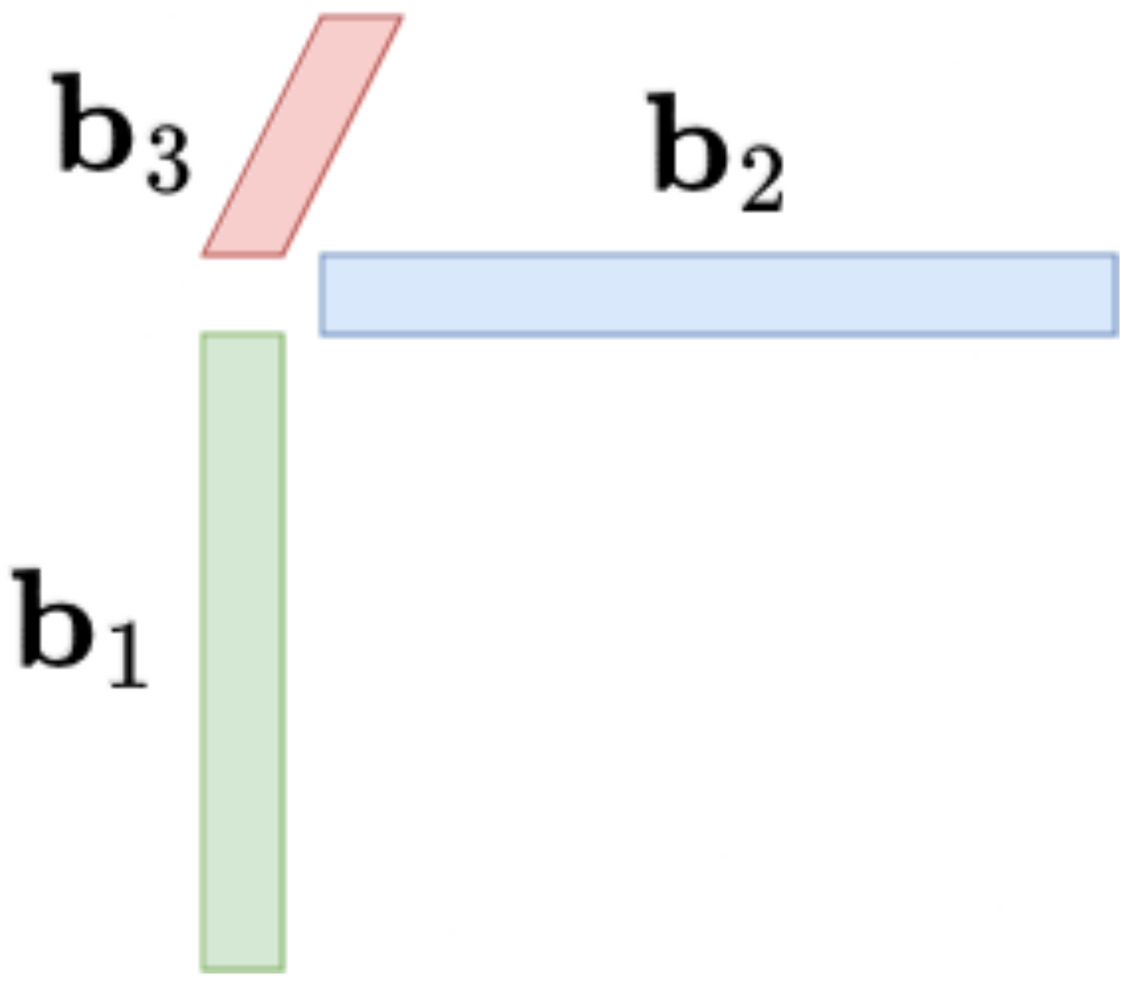
- 2D: a rank-1 *matrix*
- rank-r matrix can be written as the sum of r rank-1 matrices.

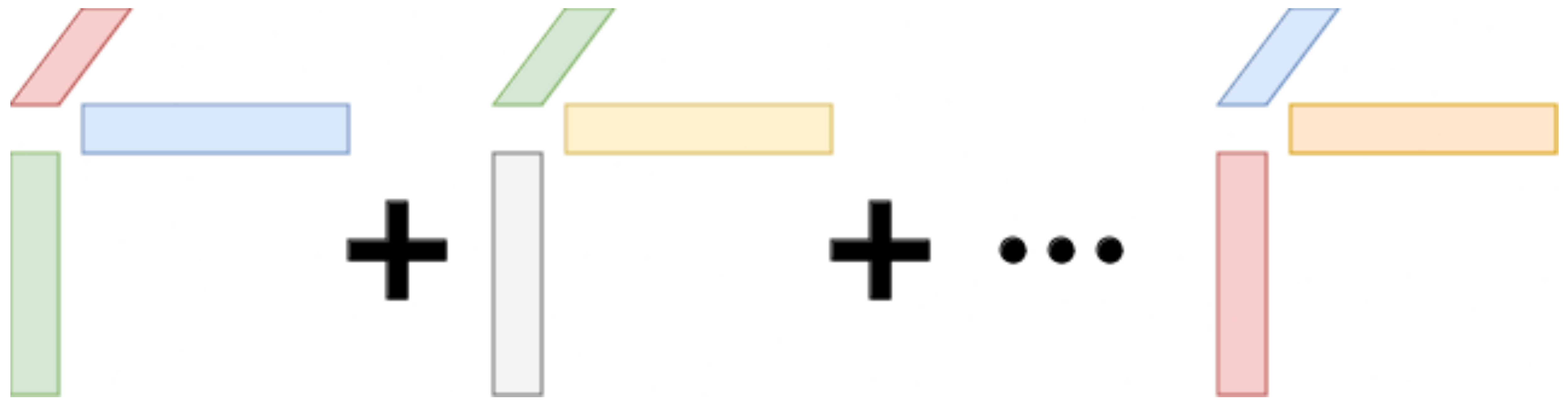


Rank-1 tensors are outer products

Trying to get a handle on rank

- 2D: a rank-1 *matrix*
- rank-r matrix can be written as the sum of r rank-1 matrices.
- A matrix has a **CANDECOMP/ PARAFAC (CP)** representation of order r if we can write it as a sum of r rank-1 outer products.

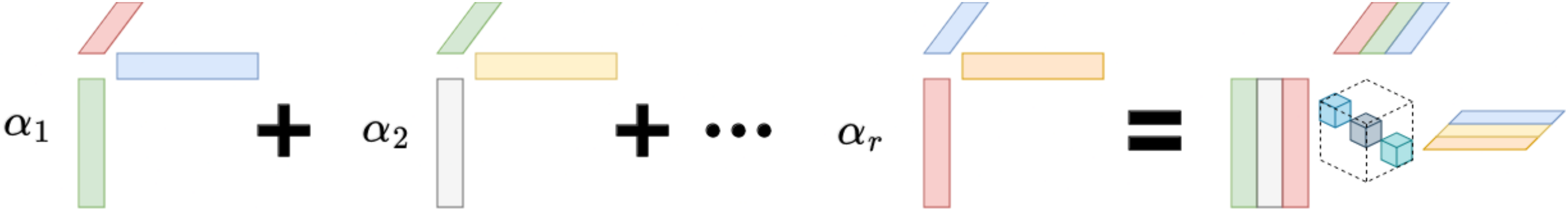




CP Decomposition

CP factorization

Writing the decomposition with matrix-tensor products



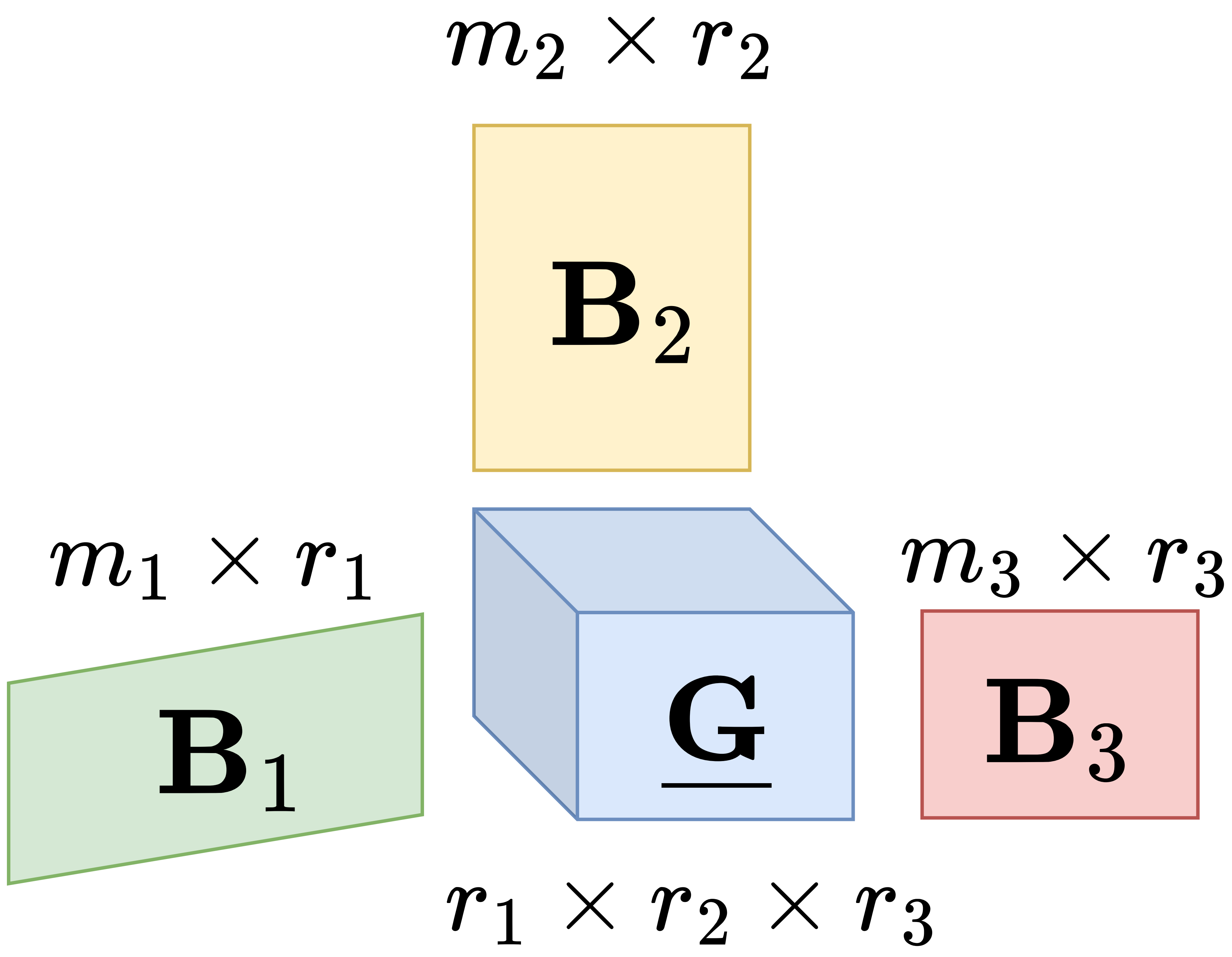
Gather the factors from each mode into matrices and define an $r \times r \times \cdots \times r$ **diagonal core tensor** $\underline{\mathbf{G}}$:

$$\underline{\mathbf{B}}_{\text{CP}} = \underline{\mathbf{G}} \times_1 \mathbf{B}_1 \times_2 \mathbf{B}_2 \cdots \times_K \mathbf{B}_K$$

The total number of parameters is $r\left(1 + \sum_{k=1}^{K} m_k\right)$ as opposed to $\prod_{k=1}^{K} m_k$.

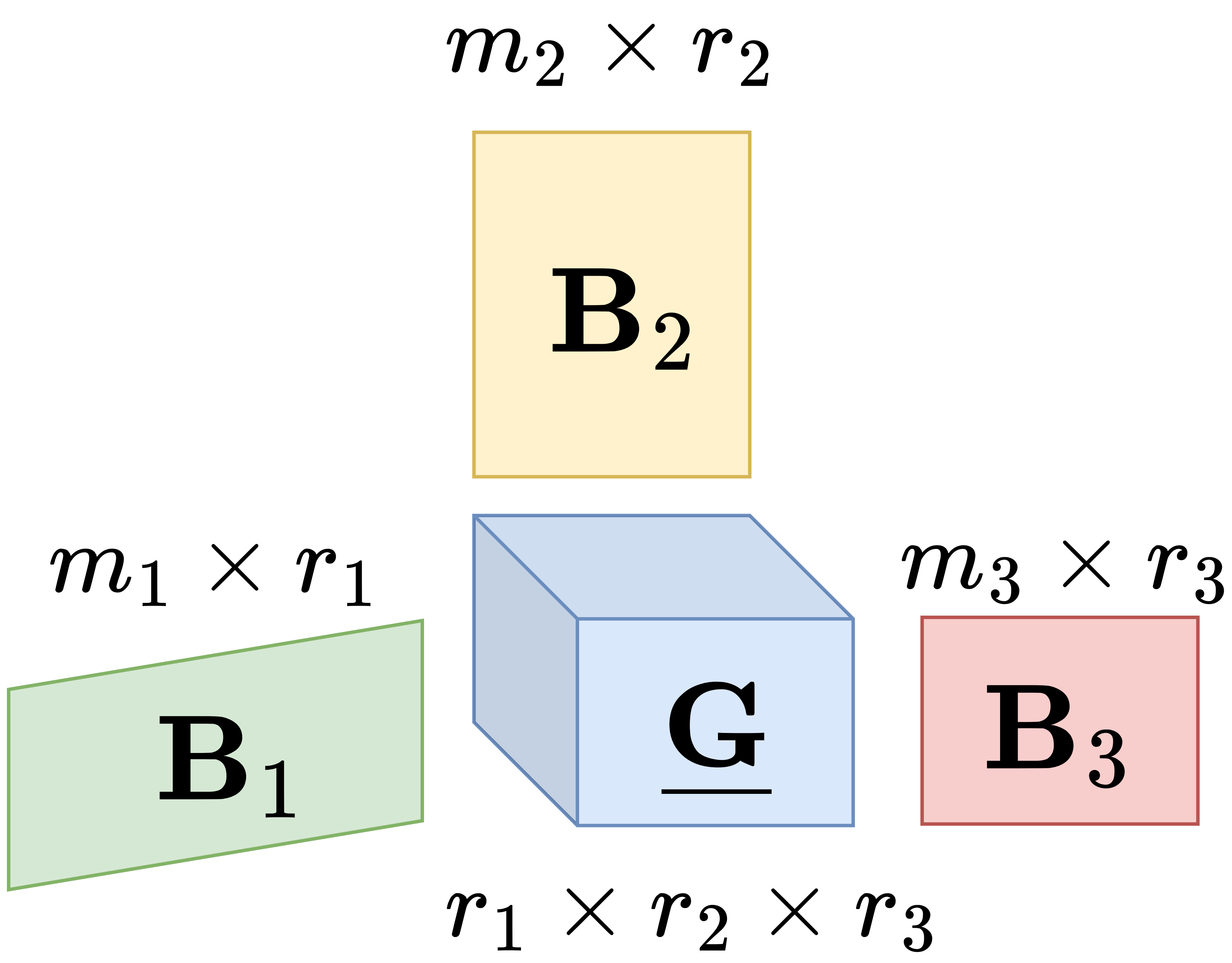
Tucker decomposition

Filling out the core tensor



Tucker decomposition

Filling out the core tensor



Suppose we have a **core tensor**

$$\underline{\mathbf{G}} \in \mathbb{R}^{r_1 \times r_2 \times \cdots \times r_K}$$

and expand the dimensions using matrix-tensor products. This is the **Tucker decomposition**:

$$\underline{\mathbf{B}}_{\text{Tucker}} = \underline{\mathbf{G}} \times_1 \mathbf{B}_1 \times_2 \mathbf{B} \times_3 \mathbf{B}_3$$

The total number of parameters is

$$\prod_{k=1}^{K} r_k + \sum_{k=1}^{K} m_k r_k$$

Other tensor decompositions

A plethora of options

There are other tensor decompositions out there (see Cichocki 2016):

- Tensor Train
- Hierarchical Tucker/Tree Tensor Network States

Our proposal is to use a simpler form of a **block tensor decomposition** (Section 5.7, Kolda and Bader 2009), which can written as a **mixture of Tucker models**:

$$\underline{\mathbf{B}}_{\text{BTD}} = \sum_{s=1}^{S} \underline{\mathbf{G}}_s \times_1 \mathbf{B}_{1,s} \times_2 \mathbf{B}_{2,s} \cdots \times_K \mathbf{B}_{K,s},$$

In general, each $\underline{\mathbf{G}}_s$ can have a different size, so we need to choose S *and* $\{m_{k,s}, r_{k,s}\}$ for each $s \in [S]$. We will assume a common $\underline{\mathbf{G}}$ for all terms.

Issues with decompositions

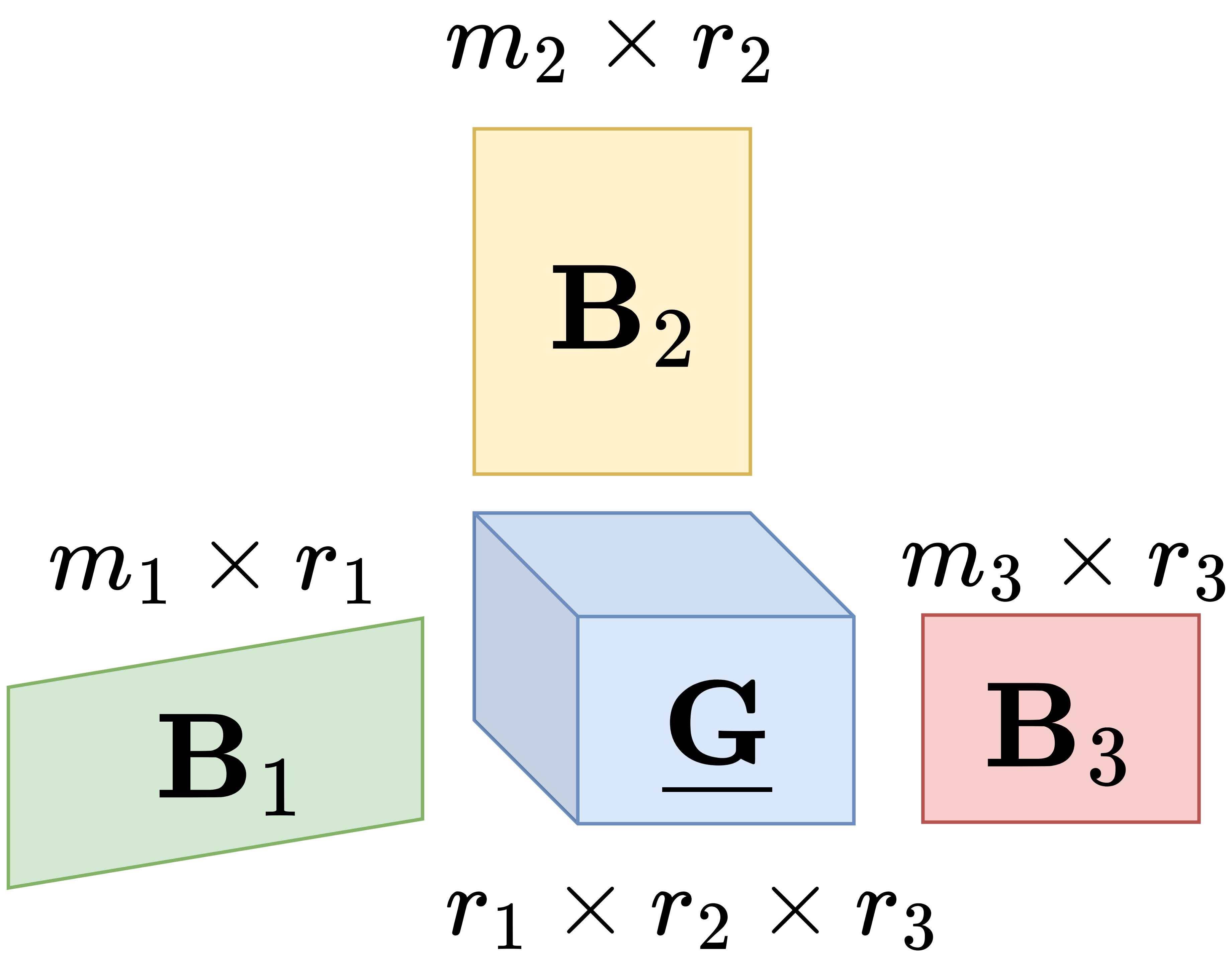
There are many different definitions of "rank" for tensors

- **CP rank** of $\underline{\mathbf{B}}$ = smallest number of terms in a CP decomposition (Hitchcock 1927, Kruskal 1977).
  - 👍🏽 The decomposition is (often) unique.
  - 👎🏽 Computing the rank is NP-complete for finite fields and NP-hard for $\mathbb{Q}$ (Håstad 1990, resolving a conjecture of Gonzalez and Ja'Ja' 1980).
- **Tucker rank** is a **vector**. Decomposition can be computed using the higher-order SVD [HOSVD] or other algorithms (De Lathauwer et al. 2000, also others).
  - Tucker rank is **not** unique.

Matrix Equivalents of Tensor Factorizations

A different kind of vectorization

Matrix-tensor products as matrix vector products



Start with a Tucker factorization:

$$\underline{\mathbf{B}}_{\text{Tucker}} = \underline{\mathbf{G}} \times_1 \mathbf{B}_1 \times_2 \mathbf{B}_2 \cdots \times_K \mathbf{B}_K$$

If we vectorzize $\underline{\mathbf{B}}_{\text{Tucker}}$, we get get the following equivalent model:

$$\text{vec}(\underline{\mathbf{B}}_{\text{Tucker}}) = \left(\mathbf{B}_K \otimes \cdots \otimes \mathbf{B}_1\right) \text{vec}(\underline{\mathbf{G}})$$

where $\otimes$ is the **Kronecker product**.

The Kronecker product

Matrix-tensor products as a matrix vector product

The Kronecker product makes "copies" of one matrix inside the other:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

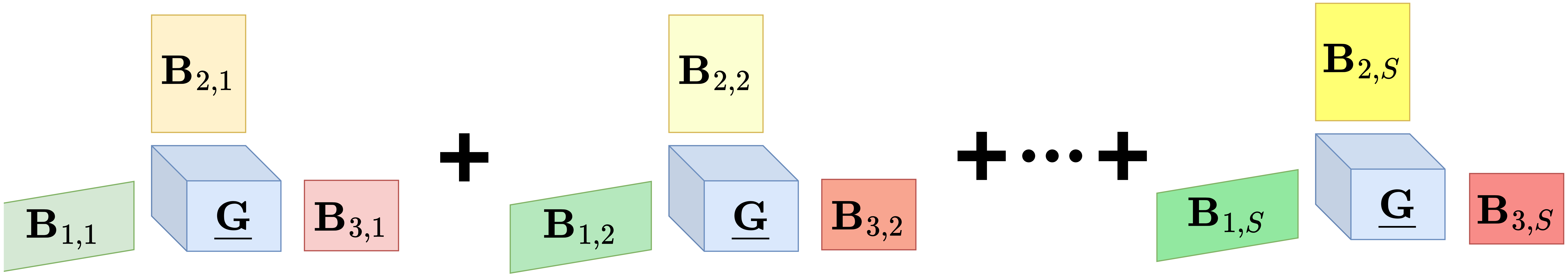
Vectorizing shows that the Tucker decomposition

$$\mathrm{vec}(\underline{\mathbf{B}}_{\text{Tucker}}) = \left(\mathbf{B}_K \otimes \cdots \otimes \mathbf{B}_2 \otimes \mathbf{B}_1\right) \mathrm{vec}(\underline{\mathbf{G}})$$

Is somewhat restrictive.

Proposal: low separation rank (LSR) tensors

BTD with a common core tensor



Special case of the BTD is a **low separation rank (LSR)** decomposition:

$$\underline{\mathbf{B}}_{\mathrm{LSR}} = \sum_{s=1}^{S} \underline{\mathbf{G}} \times_1 \mathbf{B}_{1,s} \times_2 \mathbf{B}_{2,s} \cdots \times_K \mathbf{B}_{K,s}$$

We use the *same core tensor* $\underline{\mathbf{G}}$ for each term. We also assume that the factor matrices $\{\mathbf{B}_{k,s}\}$ have orthonormal columns.

What does separation rank mean?

Writing matrices as sums of Kronecker products

The **separation rank** (Tsiligkaridis and Hero, 2013) of a matrix is the minimum number S of terms needed so that

$$\mathbf{M} = \sum_{s=1}^{S} \mathbf{A}_{K,s} \otimes \cdots \otimes \mathbf{A}_{2,s} \otimes \mathbf{A}_{1,s}$$

Our LSR model corresponds assuming the matrix-vector product has a matrix with low separation rank

$$\sum_{s=1}^{S} \underline{\mathbf{G}} \times_1 \underline{\mathbf{B}}_{1,s} \times_2 \underline{\mathbf{B}}_{2,s} \cdots \times_K \underline{\mathbf{B}}_{K,s} = \underline{\mathbf{B}}_{\mathsf{LSR}} \Longrightarrow \left(\sum_{s} \bigotimes_{k} \mathbf{B}_{\mathbf{k}} \right) \mathbf{g}$$

Prior work using CP and Tucker tensors

Generalized linear models

Prior work using CP and Tucker tensors

Generalized linear models

We look **LSR** models for **GLMs**:

Prior work using CP and Tucker tensors

Generalized linear models

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- **CP** + **logistic regression** (Tan et al., 2012)

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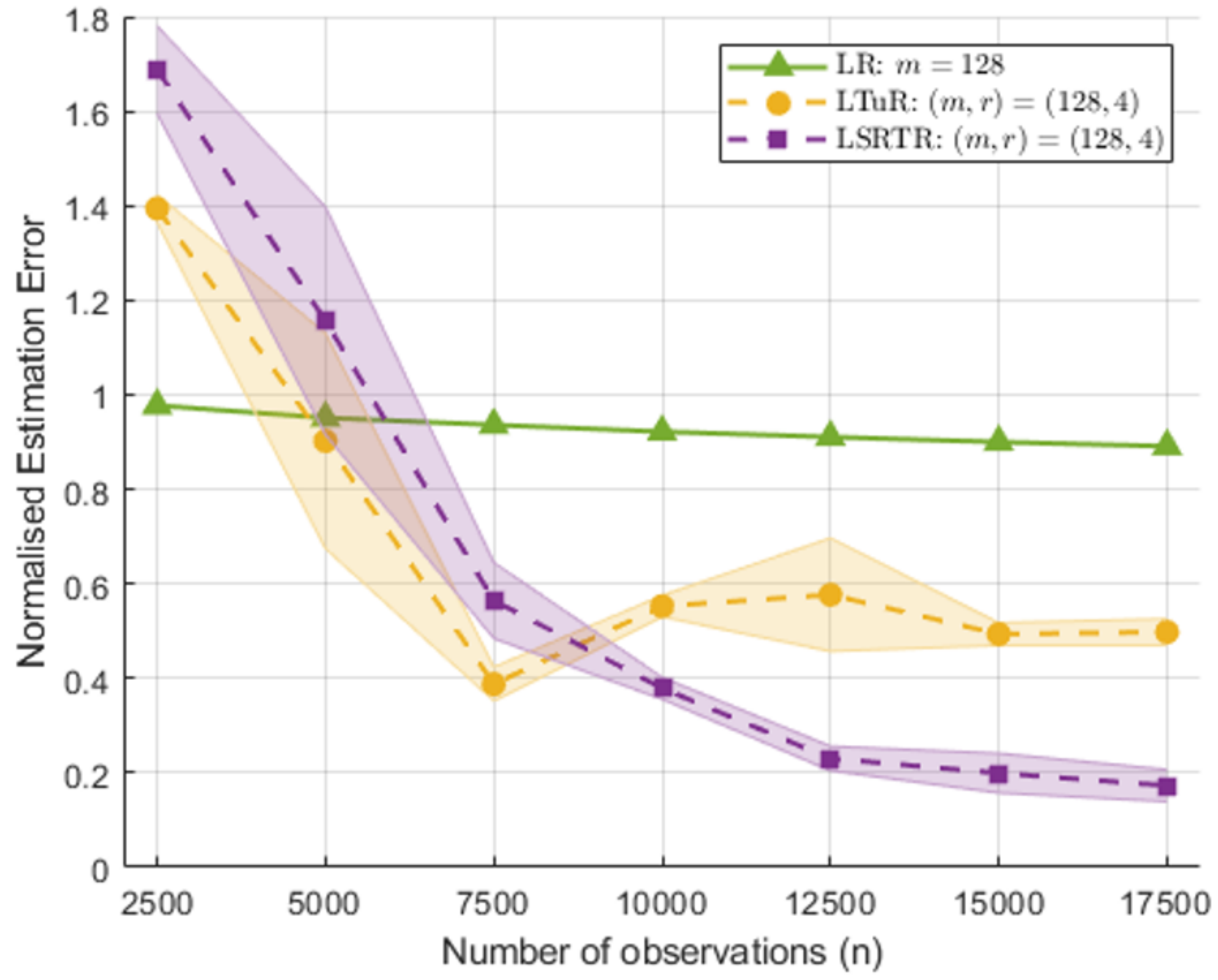
- **Tucker** + **logistic regression** (Zhang et al. 2016)
- **Tucker** + **GLMs** (Li et al., 2018; Zhou et al., 2013)

The benefits of more flexible modeling

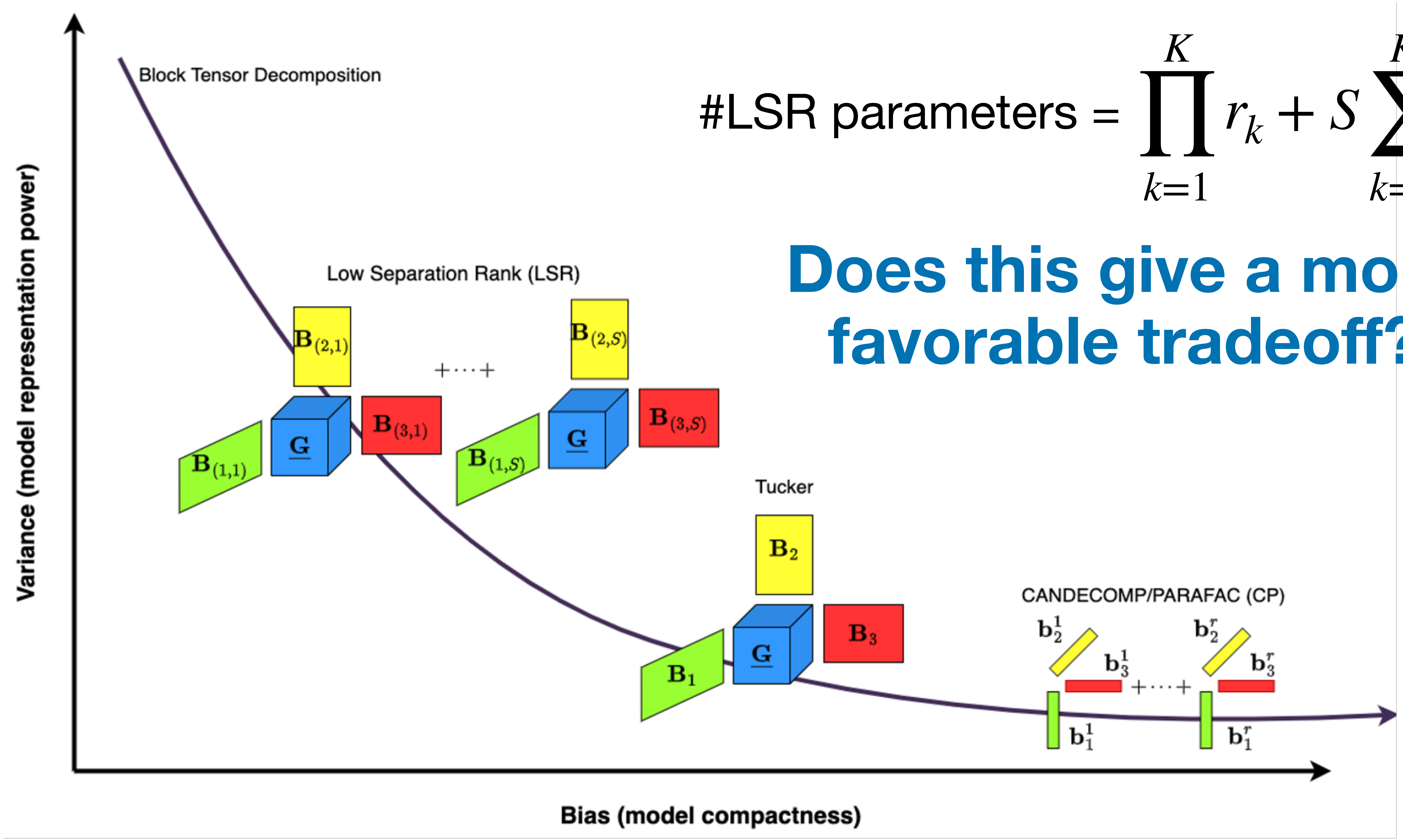
Taking advantage of more data



LSR models let use scale the number of parameters to the data set size.

Synthetic data experiments show that with a modest number of samples, LSR models are better than vectorizing or using a Tucker model.

Comparing different decompositions



$$\#\text{LSR parameters} = \prod_{k=1}^{K} r_k + S \sum_{k=1}^{K} m_k r_k$$

Does this give a more favorable tradeoff?

Regression and classification with LSR tensors

Generalized linear models for regression

Includes linear, logistic, Poisson, etc.

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We have a *training set* of n tensor-scalar pairs $\{(\underline{\mathbf{X}}_i, y_i)\}$ following a **generalized linear model (GLM)**. Model the responses y as coming from an an *exponential family*:

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Our goal: estimate $\underline{\mathbf{B}}$.

Estimation in GLMs using LSR Tensors

Mapping the tensor to a matrix

Using the LSR matrix in the vectorized problem

Mapping the tensor to a matrix

Using the LSR matrix in the vectorized problem

Under an LSR model, we have

$$\eta = \left\langle \sum_{s=1}^{S} \underline{\mathbf{G}} \times_1 \mathbf{B}_{(1,s)} \times_2 \mathbf{B}_{(2,s)} \times_3 \cdots \times_K \mathbf{B}_{(K,s)}, \underline{\mathbf{X}} \right\rangle$$

Mapping the tensor to a matrix

Using the LSR matrix in the vectorized problem

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Vectorizing:

$$\eta = \left\langle \left(\sum_{s=1}^{S} \mathbf{B}_{(K,s)} \otimes \mathbf{B}_{(K-1,s)} \otimes \cdots \otimes \mathbf{B}_{(1,s)} \right) \mathbf{g}, \mathbf{x} \right\rangle$$

Maximum likelihood estimator (MLE)

Sorry, but it's a bit messy…

The MLE comes from minimizing

$$\sum_{i=1}^{n}\left[\left\langle\left(\sum_{s=1}^{S}\bigotimes_{k}\mathbf{B}_{(k,s)}\right)\mathbf{g},\mathbf{x}_i\right\rangle T(y_i) - a\left(\left\langle\left(\sum_{s=1}^{S}\bigotimes_{k}\mathbf{B}_{(k,s)}\right)\mathbf{g},\mathbf{x}_i\right\rangle\right)\right],$$

Over all $\mathbf{B}_{k,s} \in \mathbb{O}^{m_k \times r_k}$ and $\mathbf{g} \in \mathbb{R}^{r_1 r_2 \cdots r_K}$. In practice this is not a nice optimization so we use **alternating minimization** on $\{\mathbf{B}_{(k,s)}\}$ and $\mathbf{g}$.

Question: does the MLE work and is it optimal?

Space of LSR models

Counting parameters

Suppose we are given $(r_1, r_2, \ldots, r_K, S)$. Then define

$$\mathscr{C}_{\mathsf{LSR}} = \left\{ \underline{\mathbf{B}} : \underline{\mathbf{B}} = \sum_{s=1}^{S} \underline{\mathbf{G}} \times_1 \mathbf{B}_{(1,s)} \times_2 \cdots \times_K \mathbf{B}_{(K,s)} \right\},$$

where for each (k, s), the columns of $\mathbf{B}_{(k,s)}$ are orthonormal.

Statistical/ML problems boil down to finding a “good” $\underline{\mathbf{B}} \in \mathscr{C}_{\mathsf{LSR}}$.

Question: does the # of parameters are $S \sum_k m_k r_k + \prod_k r_k$ capture the complexity?

Packing and covering LSR tensors

Statistical estimation and information theory

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Packings: find a large set of points in $\mathscr{C}_{\mathrm{LSR}}$ which are a packing in the Frobenius norm $\|\cdot\|_F$.

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Results: we get sets of the right size…

$$\approx \exp\left(S \sum_k m_k r_k + \prod_k r_k \right)$$

Identifiability using MLE

Sorry, but it's a bit messy…

Suppsse $\{(\underline{\mathbf{X}}_i, y_i) : i \in [n]\} \subset \mathbb{R}^{m_1 \times m_2 \times \cdots \times m_K} \times \mathbb{R}$ are generated from a GLM with an LSR-structured parameter $\underline{\mathbf{B}}^*$. Then if

$$n > \frac{C}{\epsilon^2}\left(\left(S\sum_k m_k r_k + \prod_k r_k\right)\log\left(\frac{C'}{\epsilon}\right) + \log\left(\frac{1}{\delta}\right)\right),$$

with probability $1-\delta$ the MLE will find a model $\hat{\underline{\mathbf{B}}}$ with excess risk no larger than ϵ.

A general lower bound for GLM + LSR

After much fun with algebra...

Suppose our data was generated with an LSR tensor $\underline{\mathbf{B}}^*$ We have a lower bound on the MSE for *any estimator* of $\underline{\mathbf{B}}^*$:

$$\mathbb{E}\left[\left\| \underline{\mathbf{B}}^* - \hat{\underline{\mathbf{B}}} \right\|_F^2\right] = \Omega\left(\frac{S\sum_k (m_k - 1)r_k + \prod_k (r_k - 1) - 1}{\left\| \mathbf{\Sigma}_x \right\|_2 n}\right)$$

We can specialize this result to the Tucker and CP cases as well.

(Taki, Sarwate, Bajwa, 2023)

Regression	Structure of $\underline{\mathbf{B}}$ Unstructured	CP	Tucker	LSR
Linear	$\frac{\sigma_y^2 \widetilde{m}}{n}$ (Raskutti et al., 2011)	–	$\frac{\sigma_y^2 \left(\sum_{k \in [K]} m_k r_k - r_k^2 + \widetilde{r} \right)}{n}$ (Zhang et al., 2020)	–
Logistic	$\frac{\widetilde{m}}{n}$ (Abramovich & Grinshtein, 2016)	–	–	–
GLM	$\frac{\sigma_y^2 \widetilde{m}}{Dn}$ (Lee & Courtade, 2020)	$\frac{\sum_{k \in [K]} m_k r + r}{M \left\Vert \mathbf{\Sigma}_x \right\Vert_2 n}$ Corollary 2	$\frac{\sum_{k \in [K]} m_k r_k + \widetilde{r}}{M \left\Vert \mathbf{\Sigma}_x \right\Vert_2 n}$ Corollary 1	$\frac{S \sum_{k \in [K]} m_k r_k + \widetilde{r}}{M \left\Vert \mathbf{\Sigma}_x \right\Vert_2 n}$ Theorem 6

Experiments and applications

Experiments on medical imaging data

Data sets and algorithms

Data sets: ABIDE Autism [fMRI] (Craddock et al., 2013 2020), Vessel MNIST 3D [MRA] (Yang et al., 2020).

Other algorithms:

- **TTR**: **Tucker** + **GLMs** using a ‘block relaxation’ algorithm (Li et al., 2018)
- **LTuR**: **Tucker** + **logistic regression** with Frobenius norm regularization (Zhang & Jiang, 2016)
- **LR**: **Unstructured** + **logistic regression** (Seber & Lee, 2003)
- **LCPR**: **CP** + **logistic regression** (Tan et al., 2013)

ABIDE Autism data set

A tiny data set: $K = 2$, $\mathbf{m} = (111{,}116)$, $n = 80$

	SVM	LR	LCPR	LTuR	LSRTR
Sensitivity	0.71	0.71	0.71	0.71	1
Specificity	0.14	0.71	0.85	0.85	0.85
F1 score	0.55	0.71	0.77	0.77	**0.93**
AUC	0.42	0.51	0.84	0.84	**0.9**
Average Accuracy	0.43	0.71	0.78	0.78	**0.92**

- Chose ranks $r_1 = 6$ and $r_2 = 6$ with $S = 2$.
- Unstructured models are quite bad in the undersampled regime.
- Adding one more Tucker component can give significant improvements.

VesselMNIST 3D

Comparing against a DNN too: $K = 3$, $\mathbf{r} = (28,28,28)$, $n = 1335$

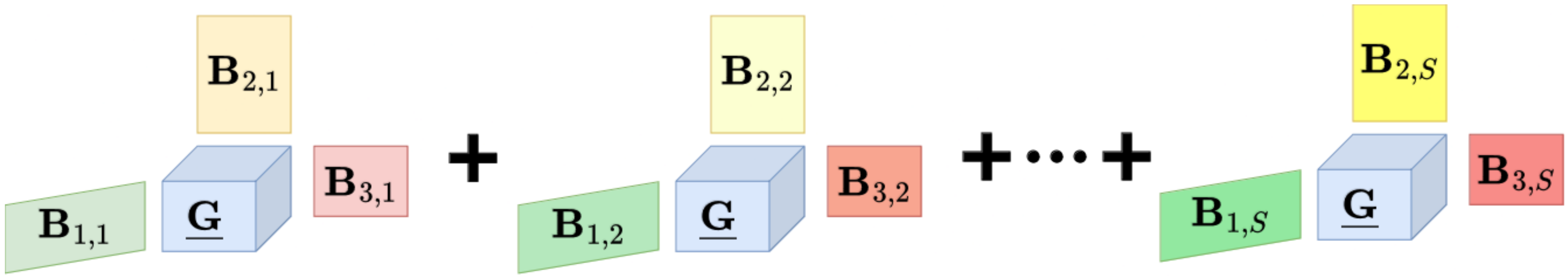
	SVM	LR	LCPR	LTuR	LSRTR	ResNet 50 + 3D
Sensitivity	0.39	0.53	0.26	0.32	0.47	0.85
Specificity	0.95	0.55	0.946	0.94	0.96	0.86
F1 score	0.44	0.21	0.3	0.37	0.55	**0.57**
AUC	0.84	0.52	0.6	0.66	0.81	**0.9**
Average Accuracy	0.89	0.55	0.869	0.87	**0.91**	0.85

- Chose ranks $r_1 = 3$, $r_2 = 3$, $r_3 = 3$, and $S = 2$
- LSRTR has better accuracy but worse F1 and AUC (see paper).
- Issues such as overfitting, interpretability, etc. are still open.

Recap and looking forward

Recap of what we've seen

Structuring tensors using factorizations for simpler modeling



There is a whole continuum of tensor decompositions and **LSR structured tensors** can be very useful:

- Adapt parameterization to the data available.
- Efficiently (empirically) learnable/estimatable.

Other uses for LSR structures

Some past, current, and ongoing directions

- Dictionary learning: theory and algorithms

$$\underbrace{\underline{\mathbf{Y}}}_{\in \mathbb{R}^{m_1 \times \cdots \times m_N}} = \sum_{s=1}^{S} \overbrace{\underbrace{\underline{\mathbf{X}}}_{\in \mathbb{R}^{p_1 \times \cdots \times p_N}}}^{\text{Sparse}} \times_1 \underbrace{\mathbf{D}_{1,s}}_{\in \mathbb{R}^{m_1 \times p_1}} \times_2 \cdots \times_N \underbrace{\mathbf{D}_{K,s}}_{\in \mathbb{R}^{m_K \times p_K}} + \underline{\mathbf{W}}$$

- Federated learning: applications in MRI
- Structuring latent space representations for generative models
- Reducing training and compute time

Even a KS assumption can help

Even better results with LSR models (S > 1)

Original Image | Noisy Image | Unstructured DL: 147456 parameters | Separable DL: 265 parameters

Many questions remain!

Lots to understand on the theory and practical side

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Lots to understand on the theory and practical side

Theory

- Algorithms for computing decompositions with good guarantees for approximation and denoising.
- Convex relaxations of LSR constraint for optimization (we have some for dictionary learning!)
- Random tensor theory and spectral analysis.

Many questions remain!

Lots to understand on the theory and practical side

Theory

- Algorithms for computing decompositions with good guarantees for approximation and denoising.
- Convex relaxations of LSR constraint for optimization (we have some for dictionary learning!)
- Random tensor theory and spectral analysis.

Practice

- More “real” applications in neuroimaging and other domains.
- Other data domains: hyperspectral imaging, chemometrics, etc.
- Information theoretic modeling.

Thank you!