An information theorist's tour of differential privacy

Anand D. Sarwate, Rutgers University

4 August 2025

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Enoshima in Sagami Province (相州江の島)

from Thirty-six views of Mount Fuji

Supported by the
IEEE Information Theory Society
Distinguished Lecturer Program

Macquarie University
Sydney, Australia

Some thanks and credits





Thanks for helpful discussions with

Shahab Asoodeh (McMaster)
Flavio Calmon (Harvard)
Oliver Kosut (Arizona State)
Lalitha Sankar (Arizona State)
Mario Diaz (UNAM) - in memoriam

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- Describe some of these three connections for those less familiar
- Suggest some questions for discussion later?

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Let's start simple

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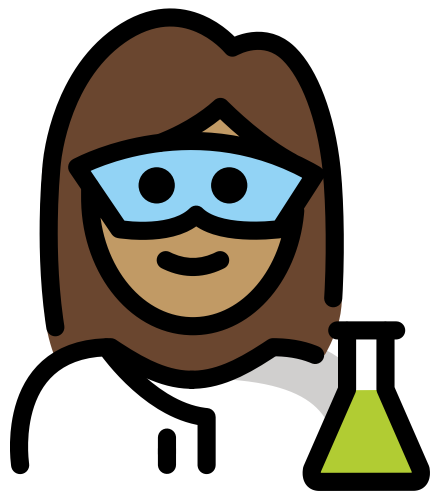
Sasha



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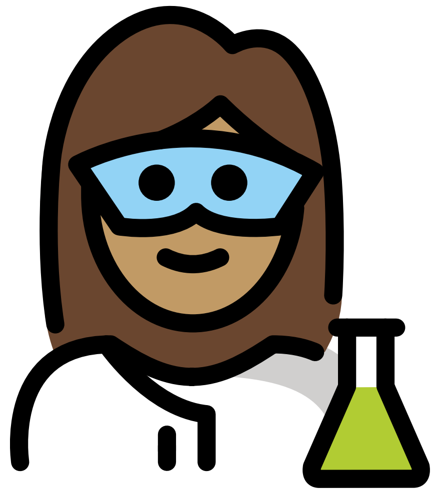

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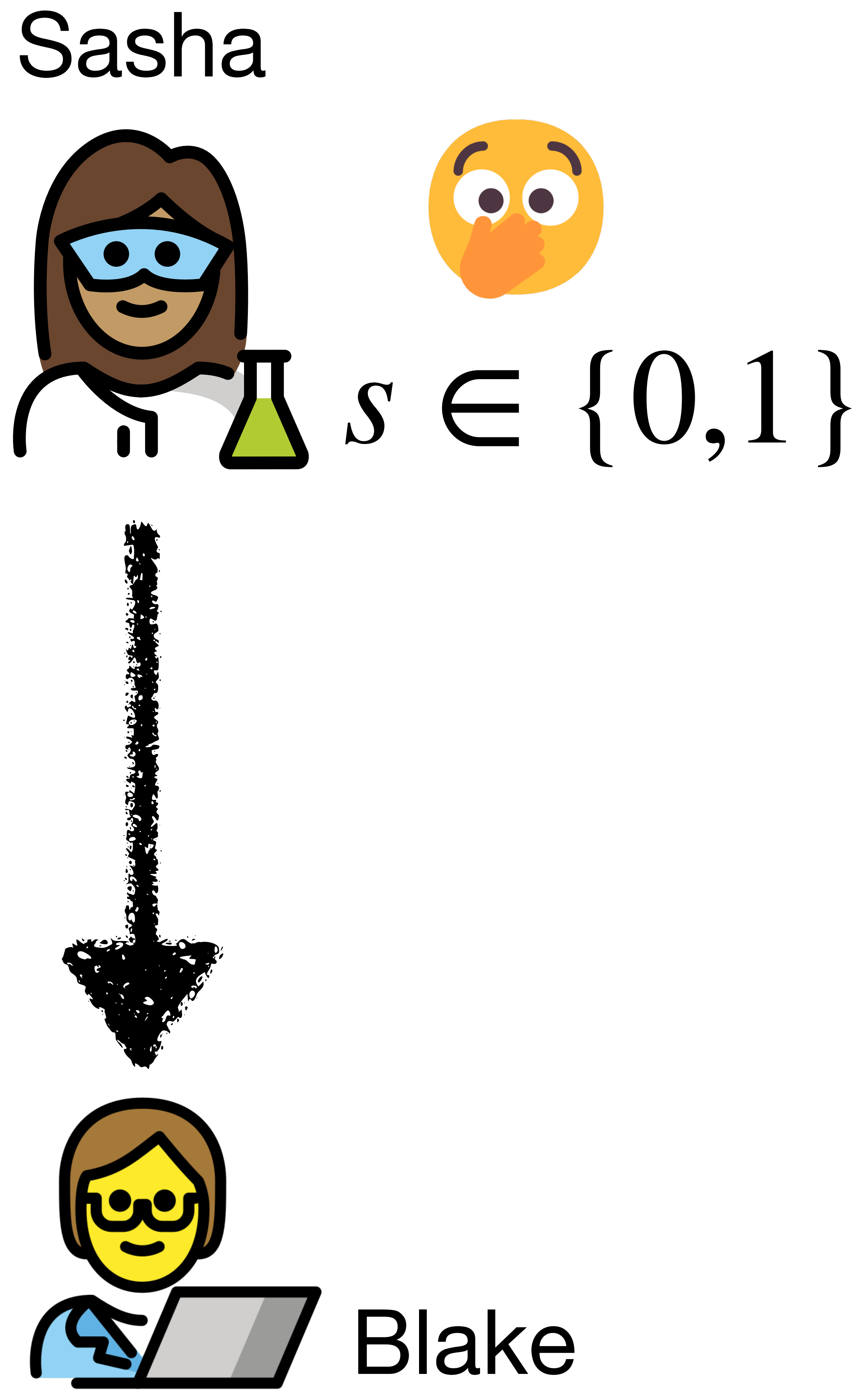

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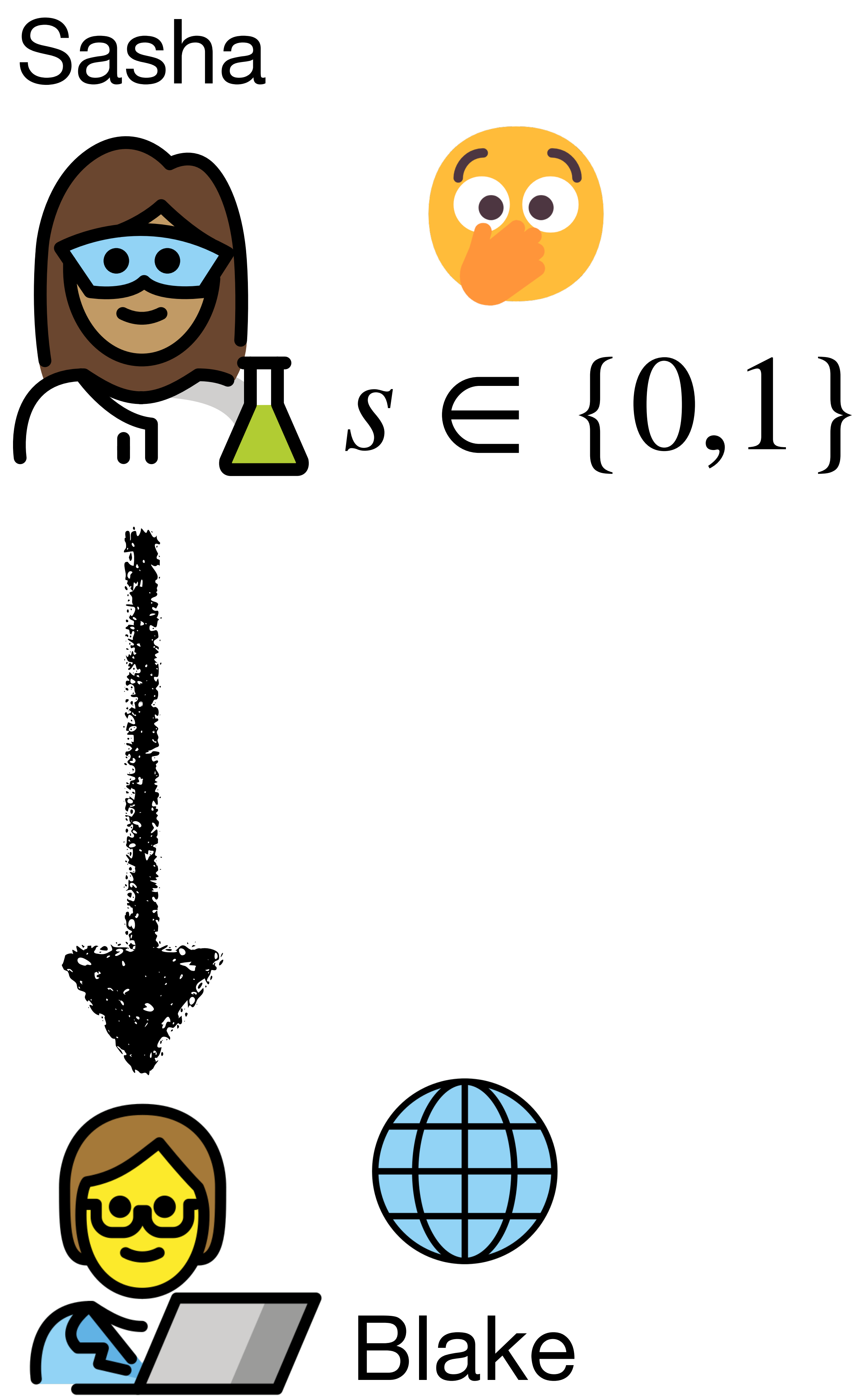

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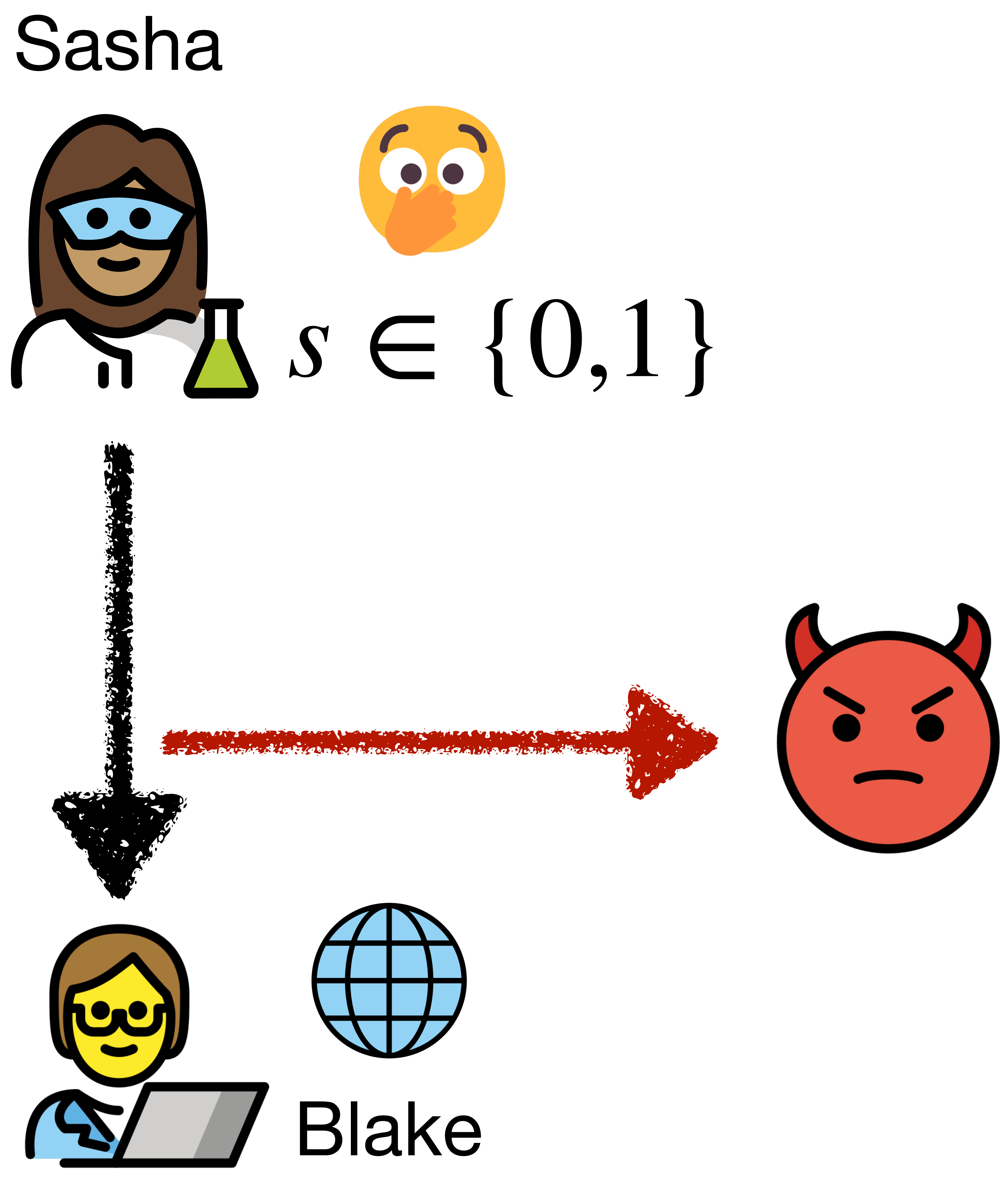

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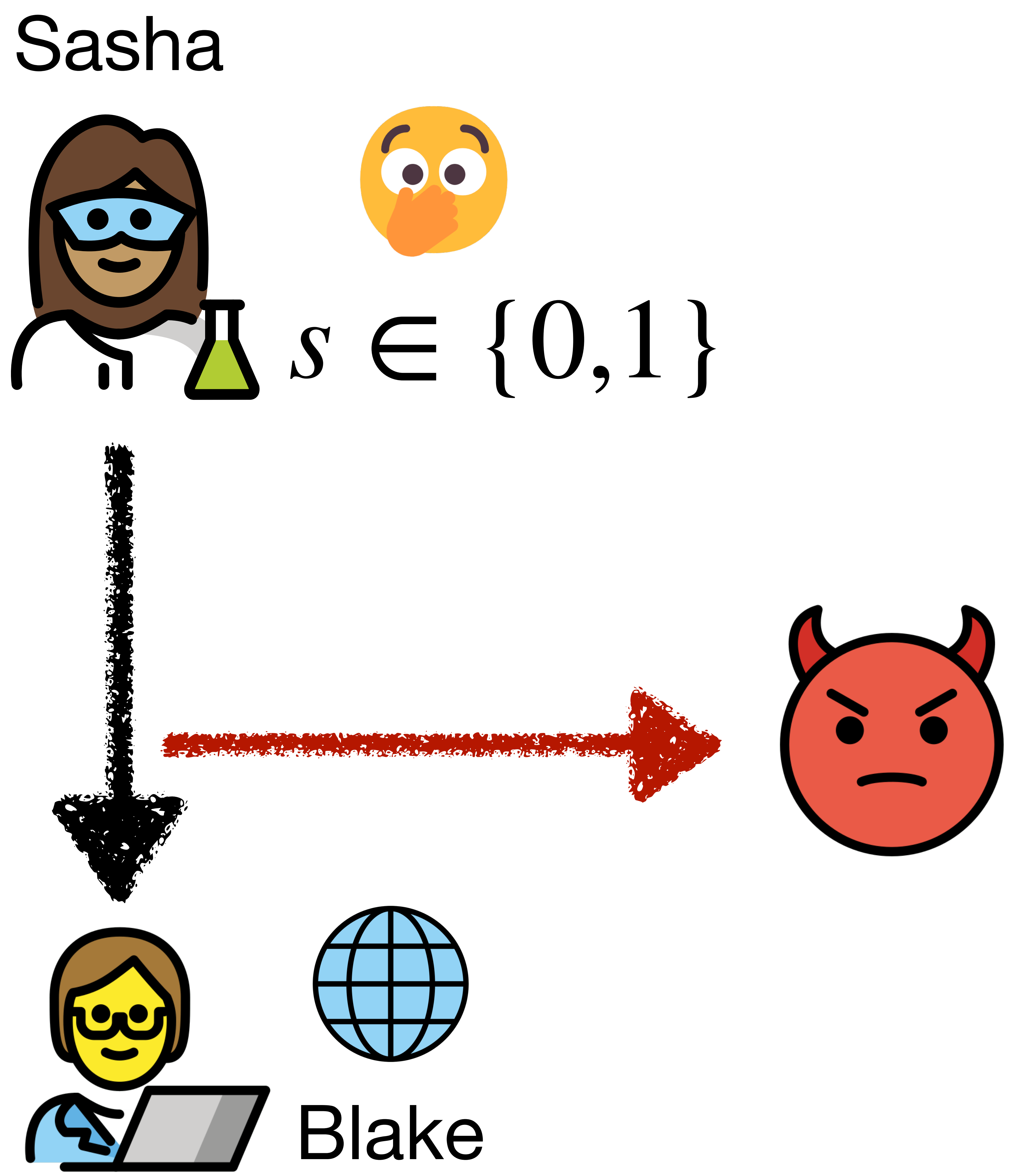

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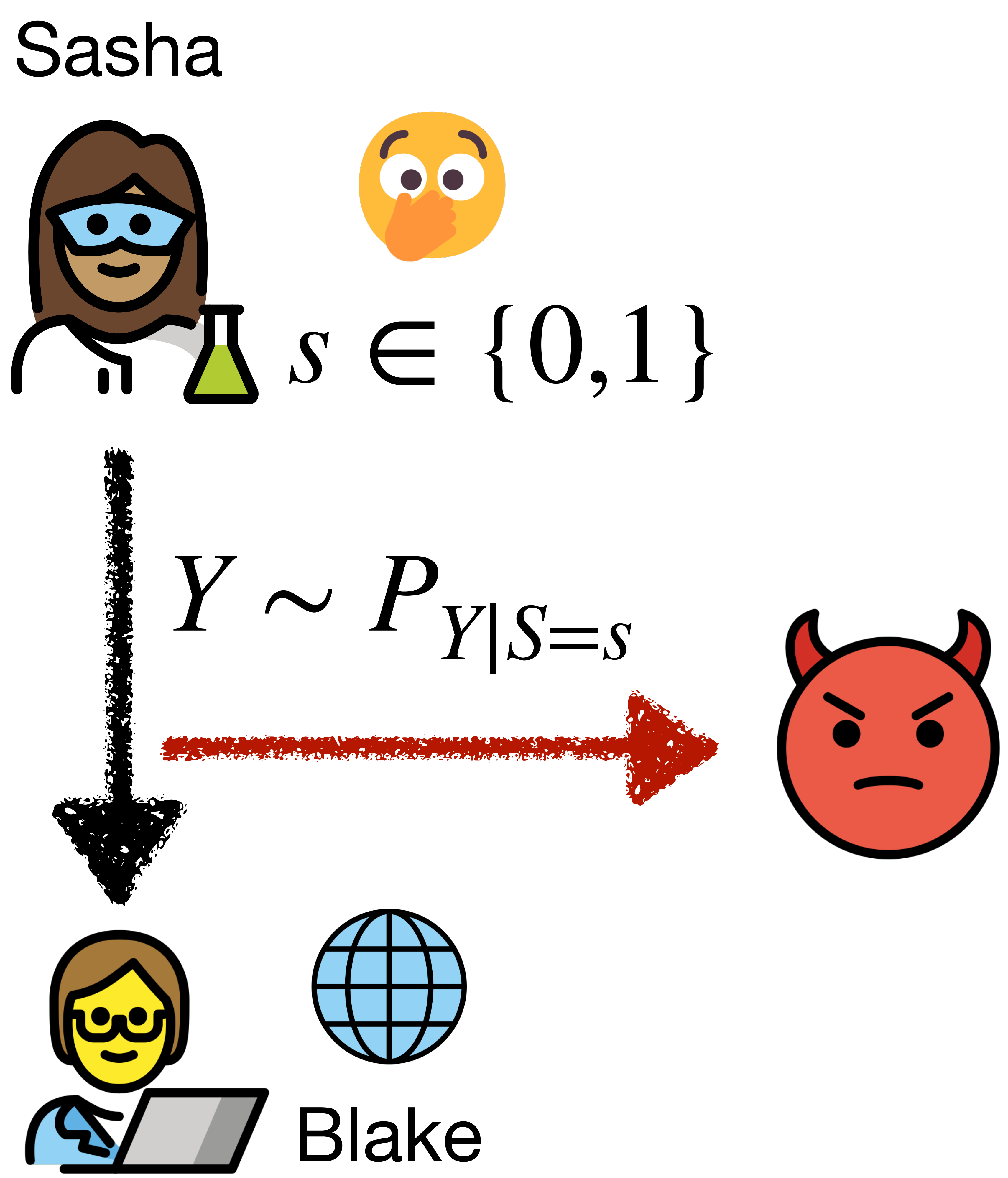

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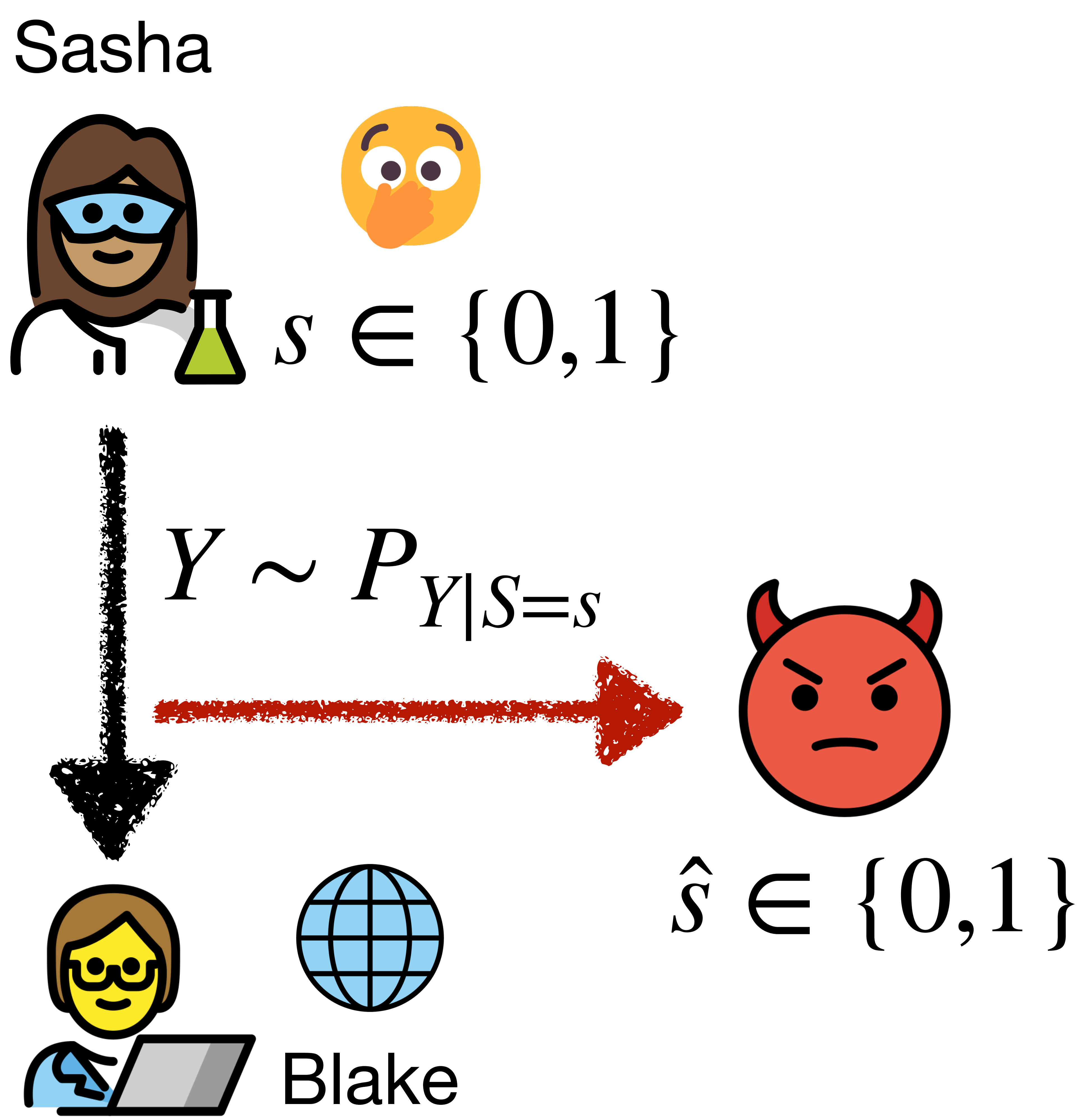

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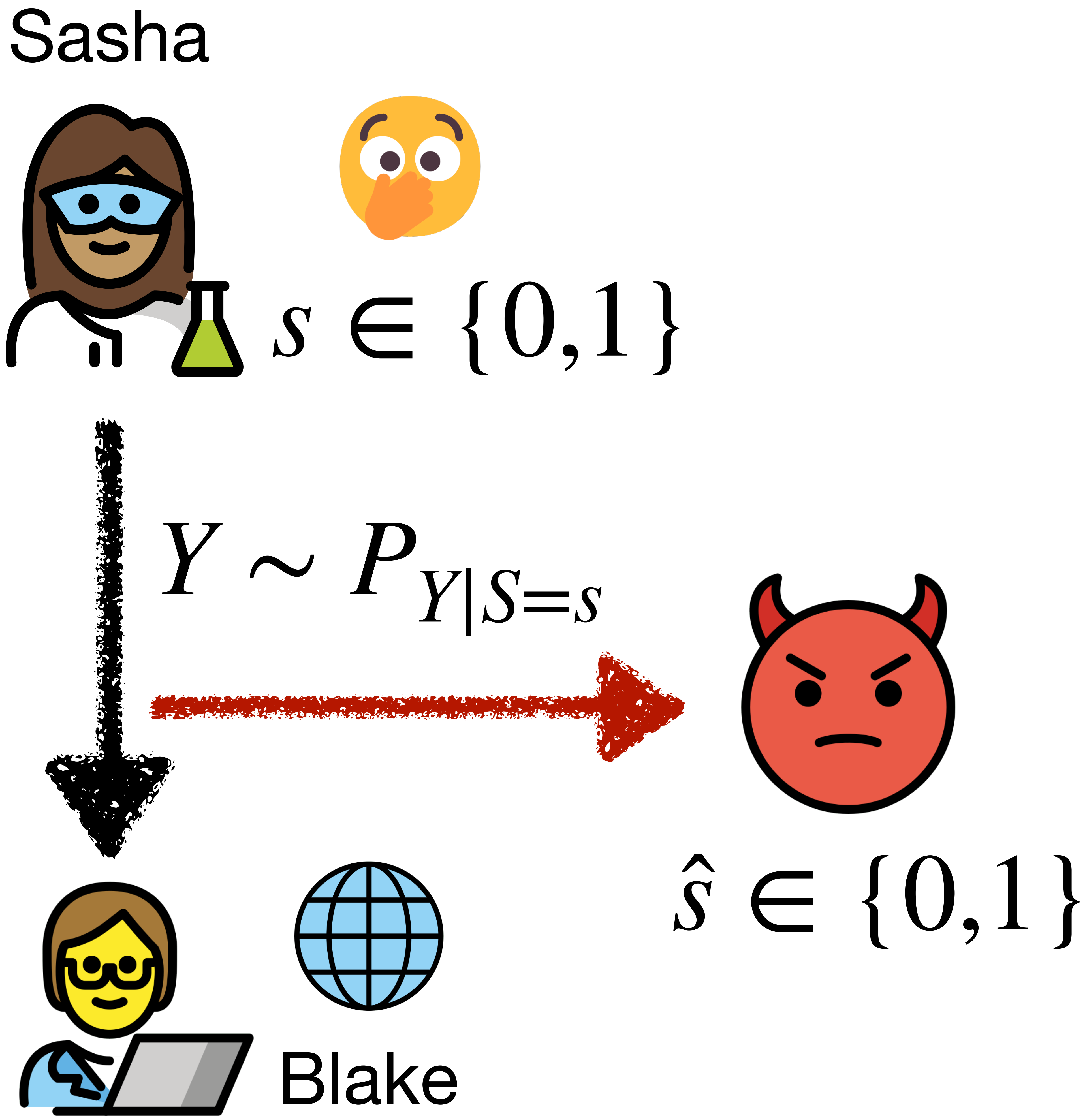

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The **privacy question** is a **hypothesis testing question**:

$$\mathscr{H}_0 : Y \sim P_{Y|S=0}$$

$$\mathscr{H}_1 : Y \sim P_{Y|S=1}$$

The Lake of Hakone in Sagami Province

相州箱根湖水

Sōshū Hakone Kosui

Vista 1

hypothesis testing

Neyman-Pearson tells us the optimal test

Adversarial inference is a generalized LRT

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The optimal test for the adversary is a **likelihood ratio test**:

$$\hat{s}(y) = \begin{cases} 1 & \log \frac{P_{Y|S=1}(y)}{P_{Y|S=0}(y)} \geq \tau \\ 0 & \log \frac{P_{Y|S=1}(y)}{P_{Y|S=0}(y)} < \tau \end{cases}$$

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Example

$\mathscr{H}_0 : Y = 0 + Z \sim \mathcal{N}(0, \sigma^2)$

$\mathscr{H}_1 : Y = 1 + Z \sim \mathcal{N}(1, \sigma^2)$

Example: additive Gaussian noise

Everyone's favorite example: Gaussians!

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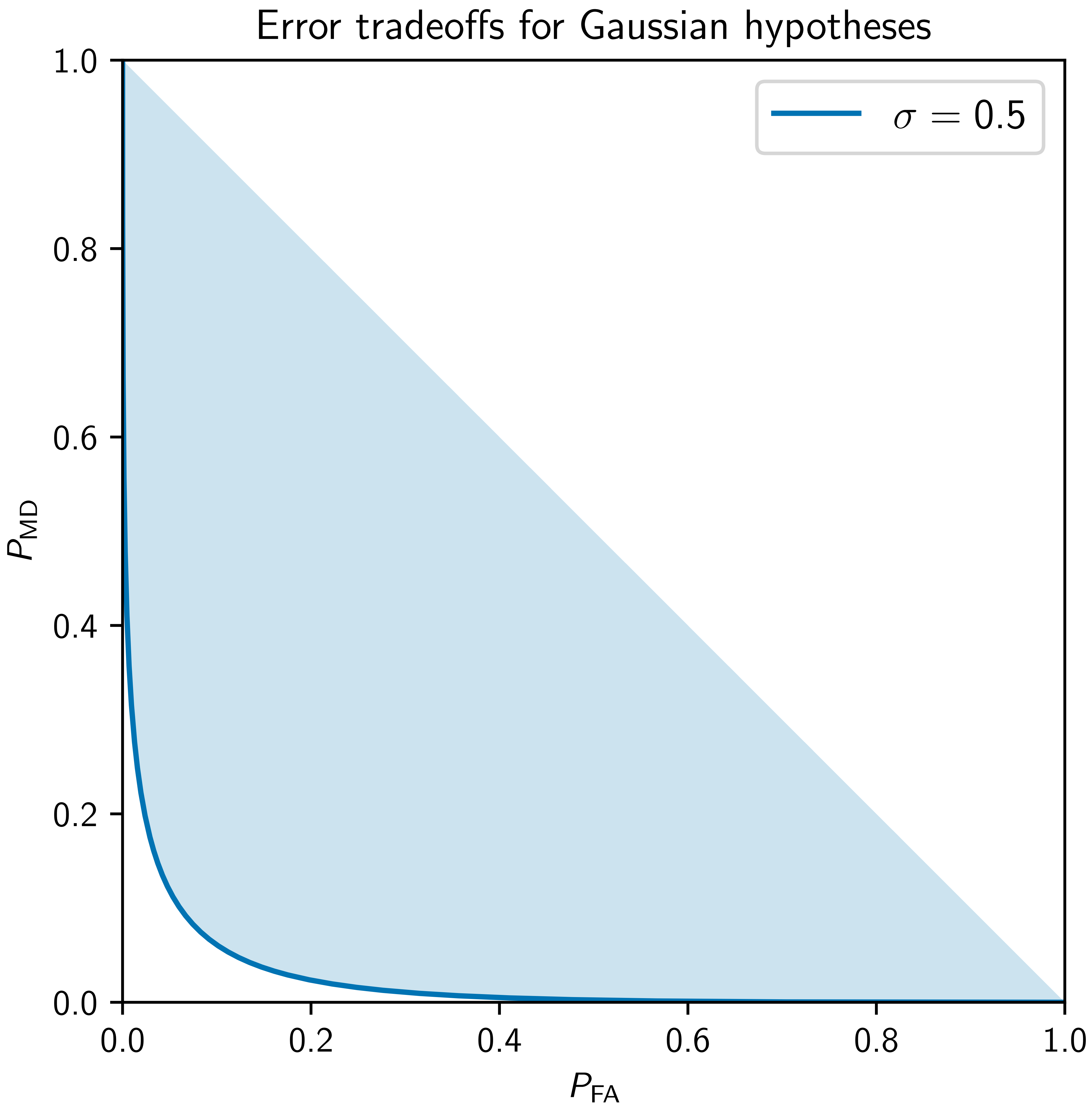

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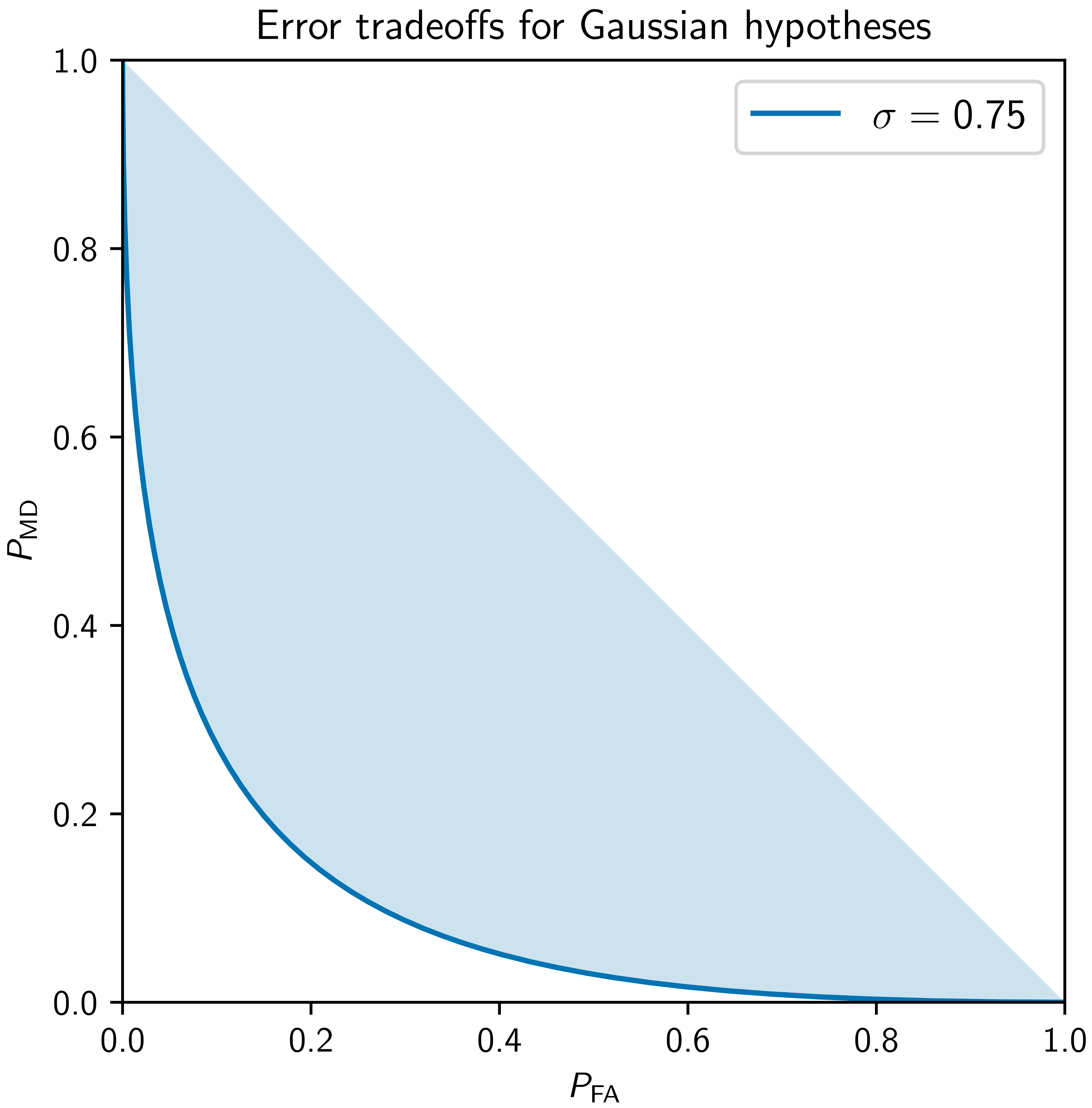

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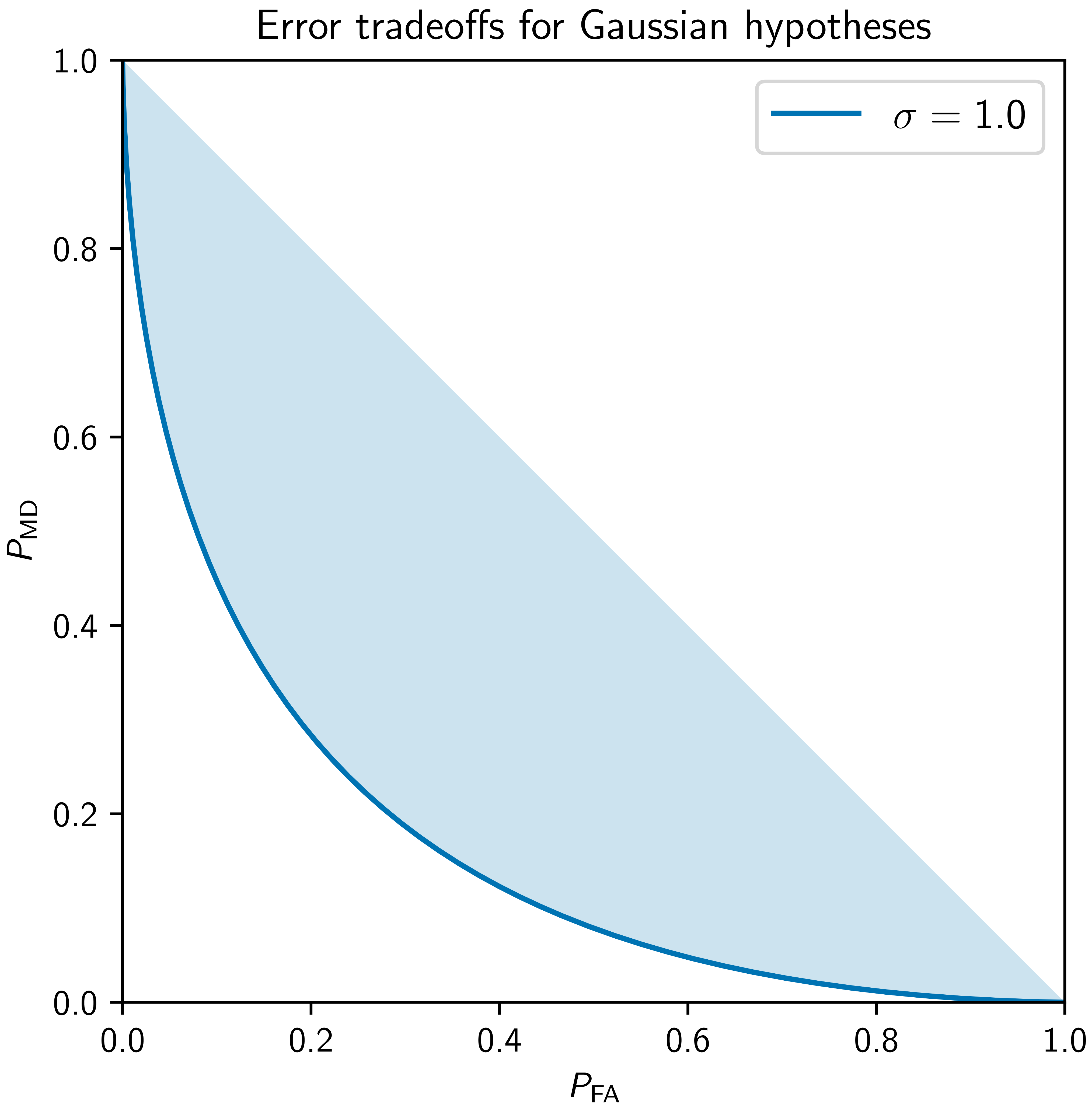

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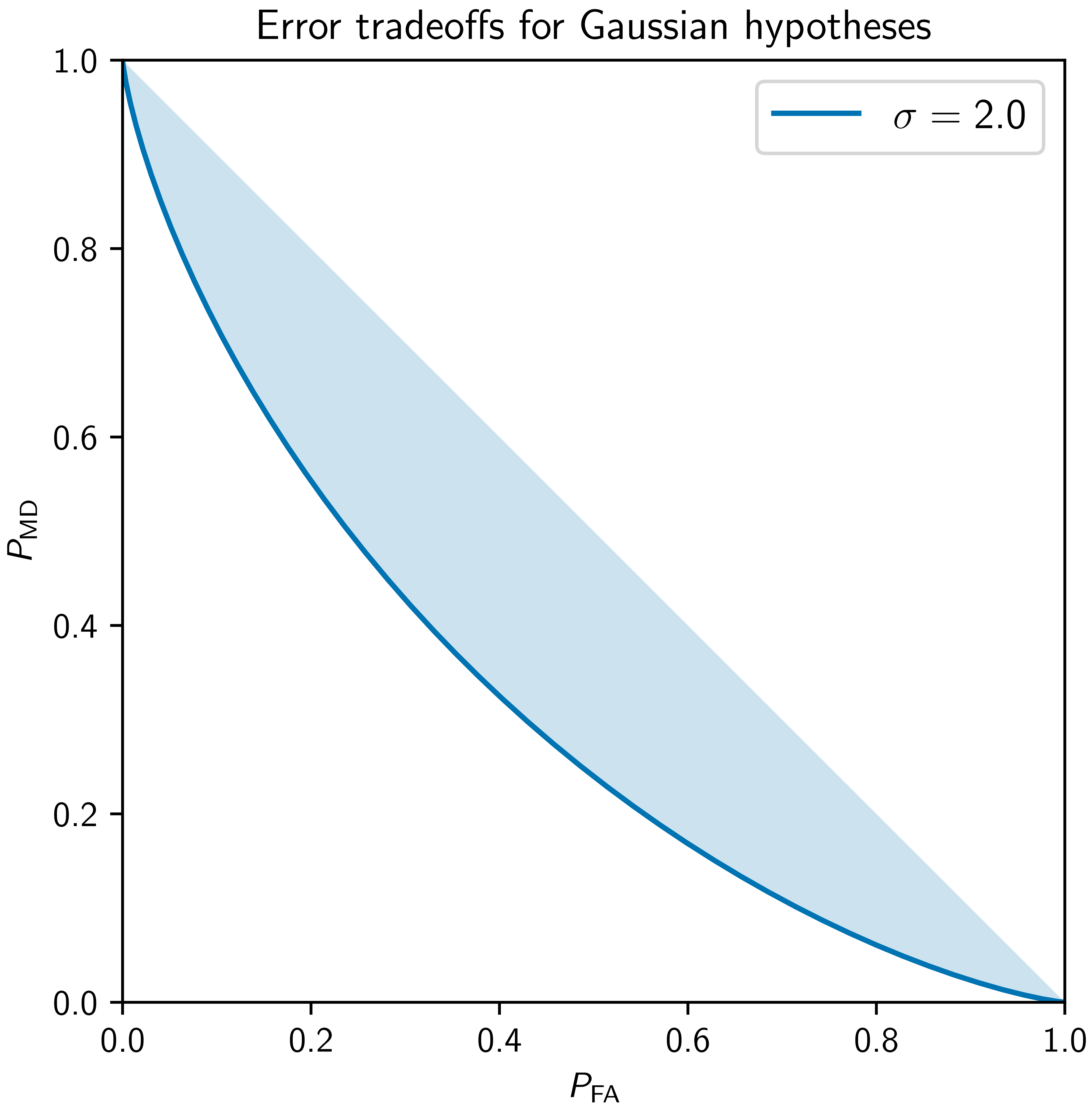

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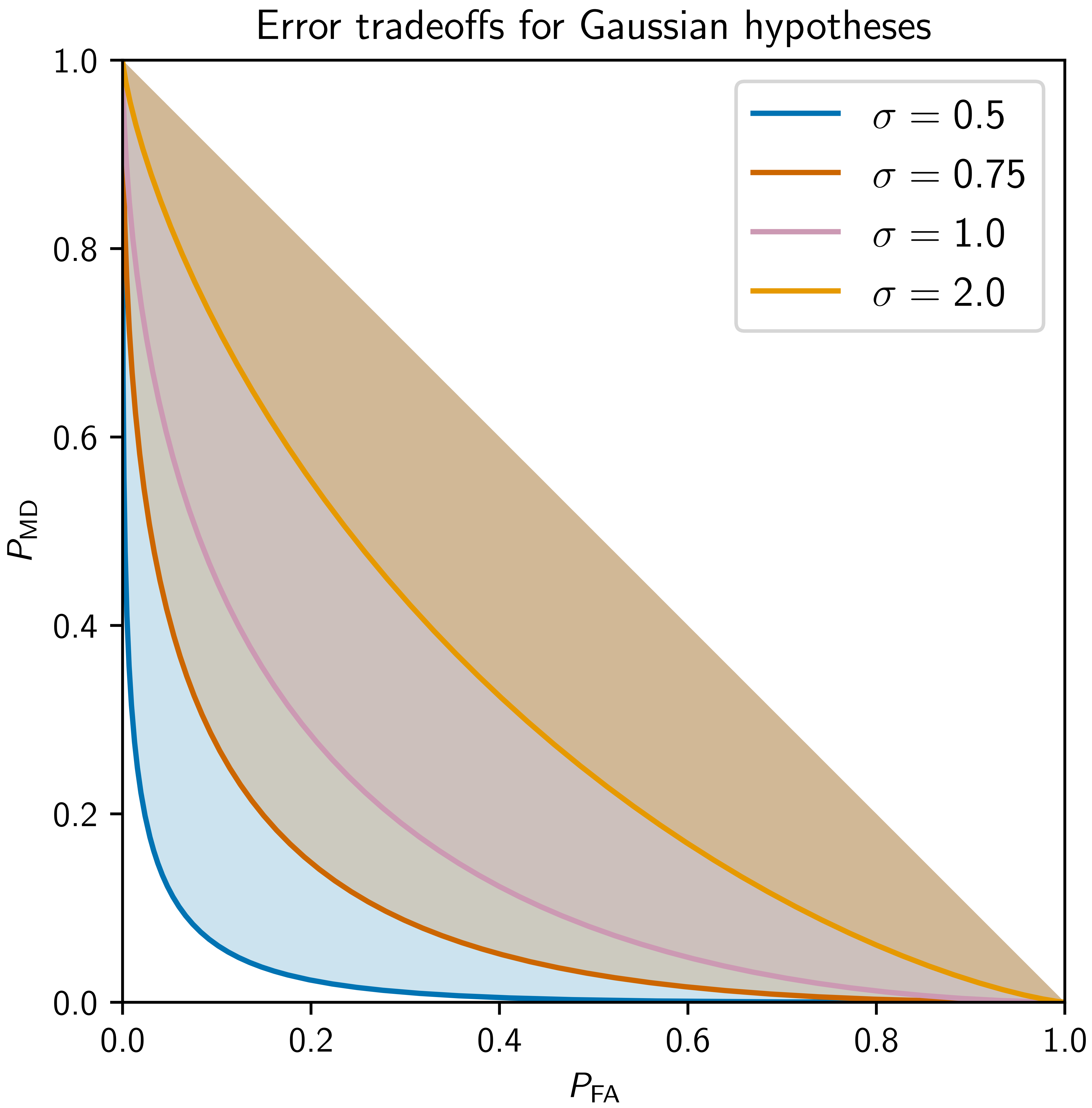

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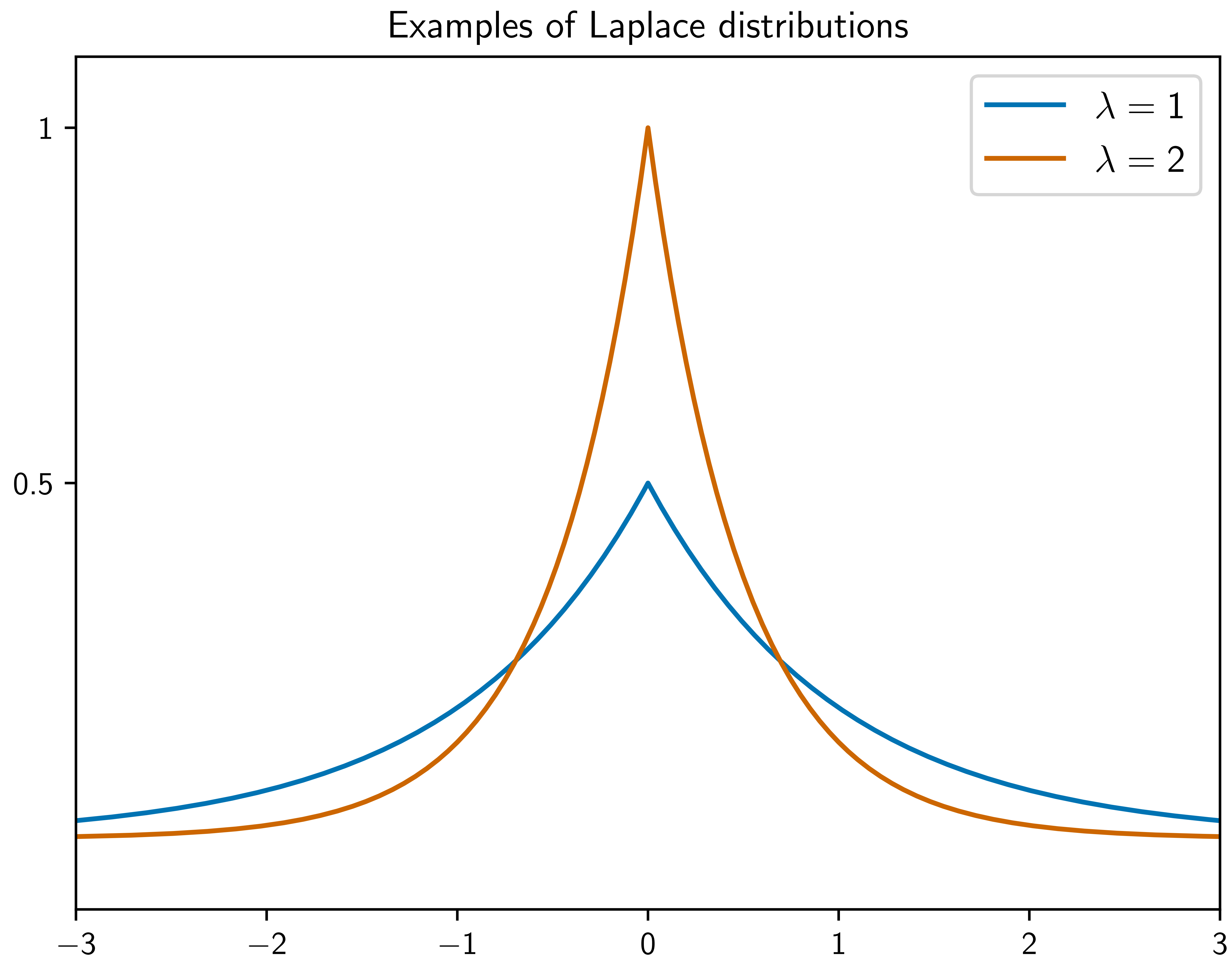

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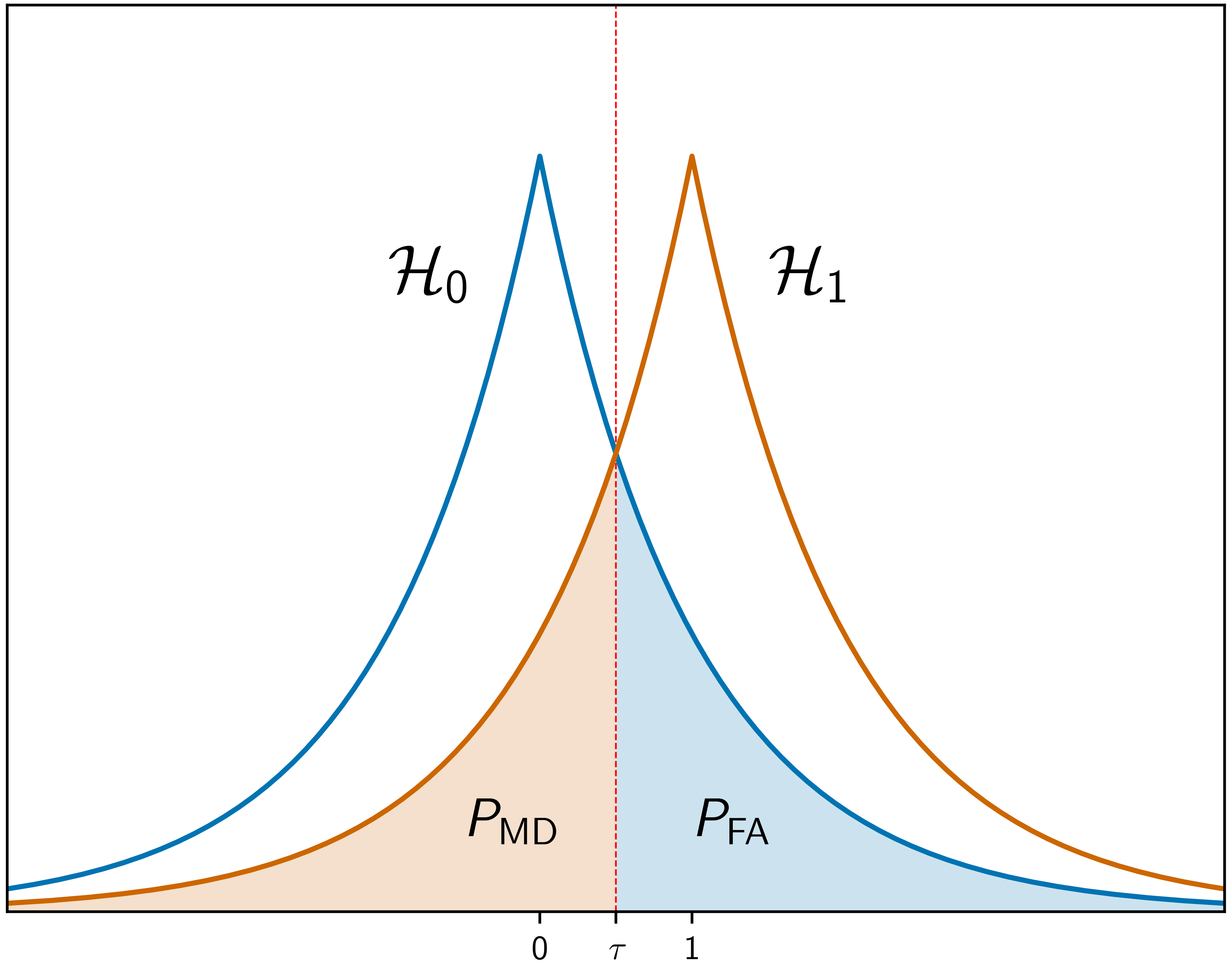

The error probabilities for the test are:

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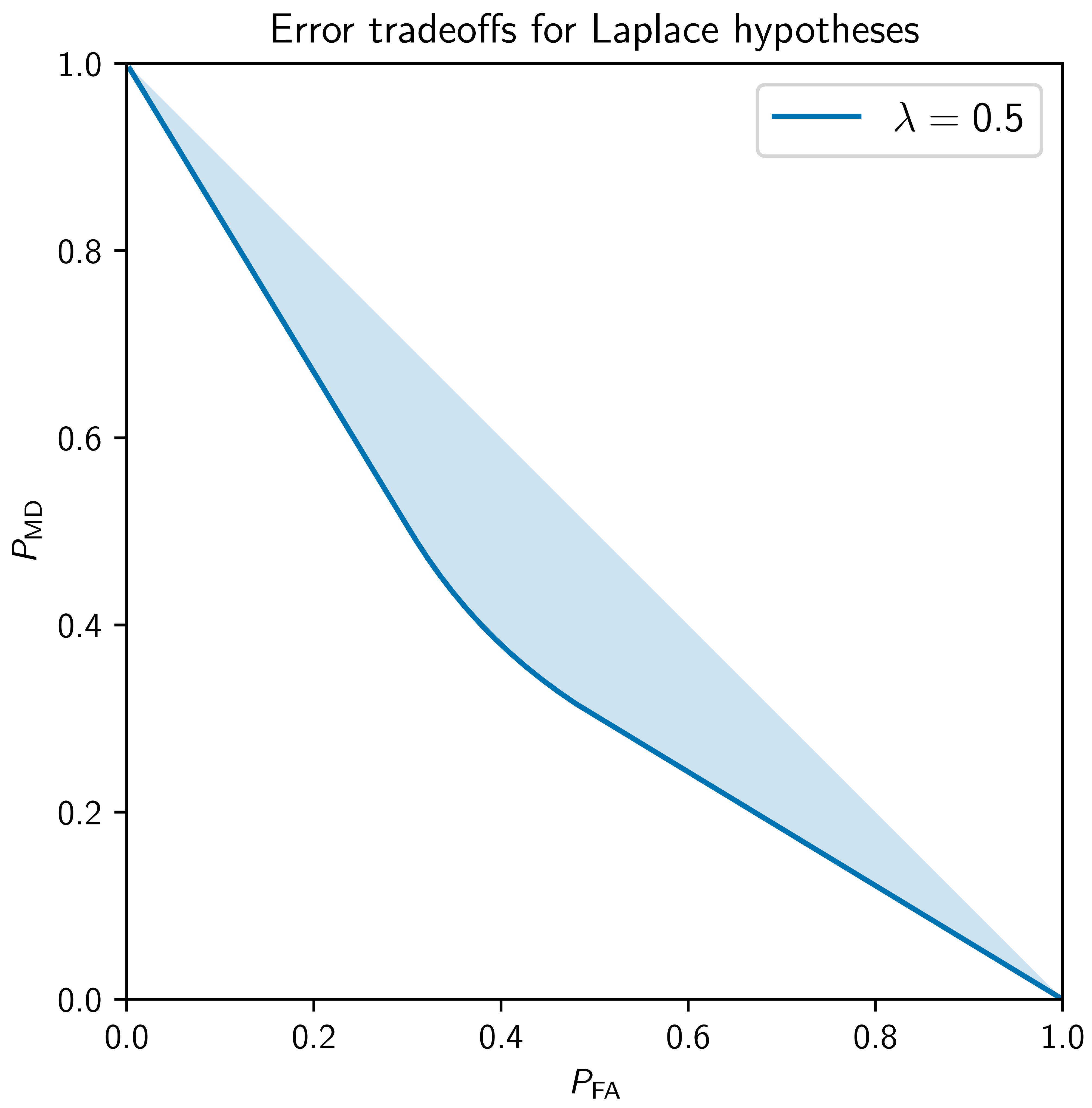

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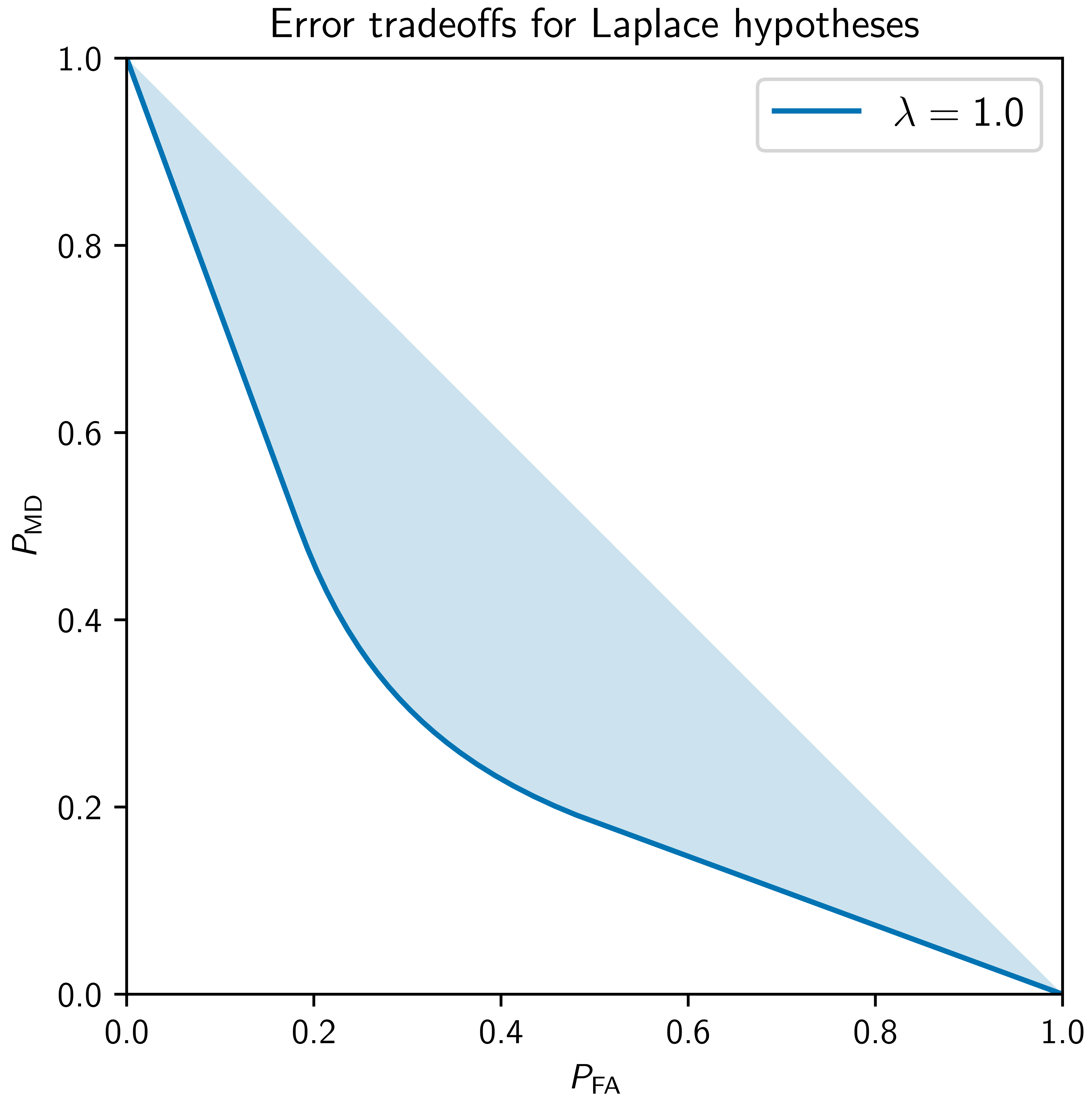

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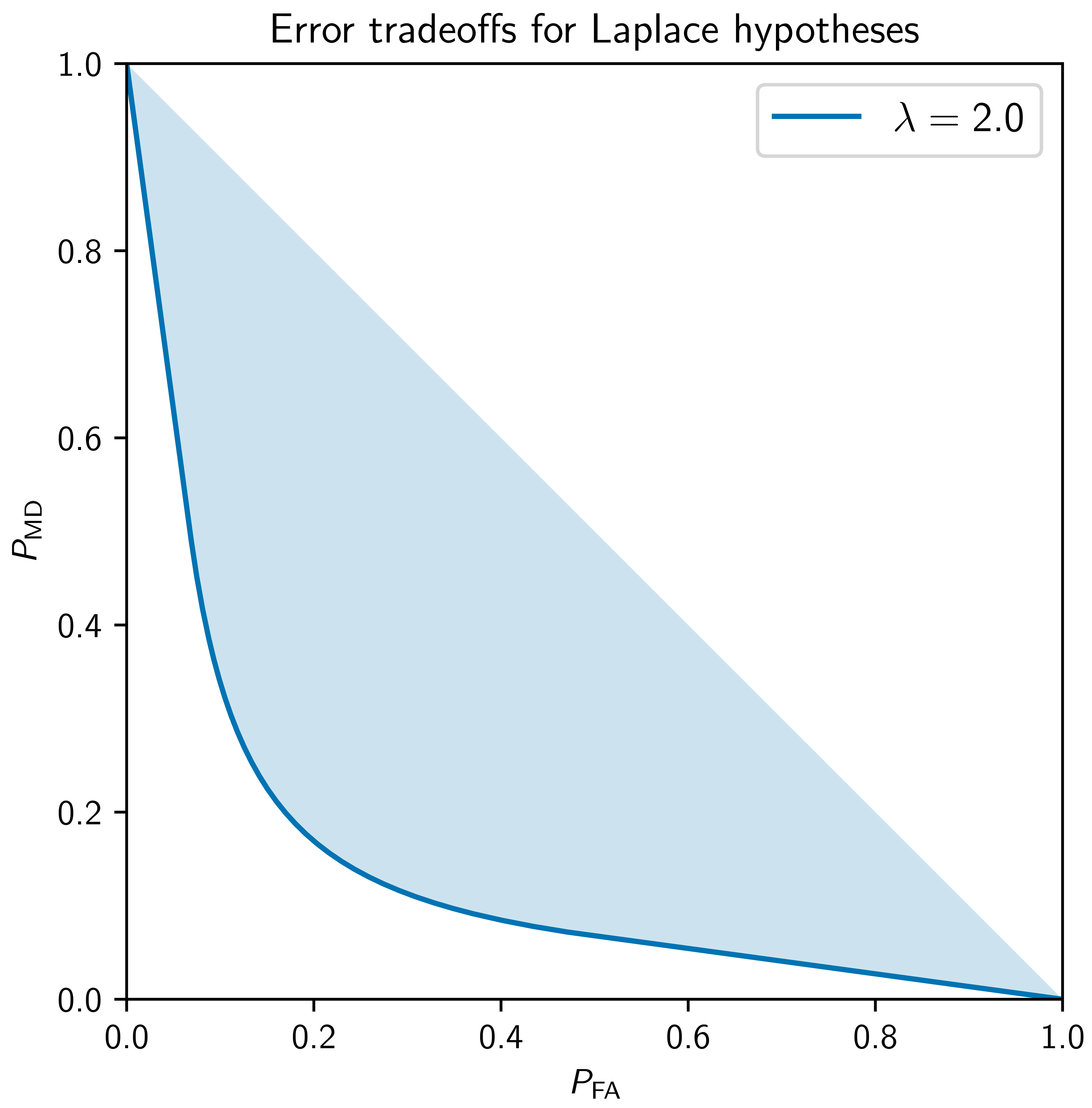

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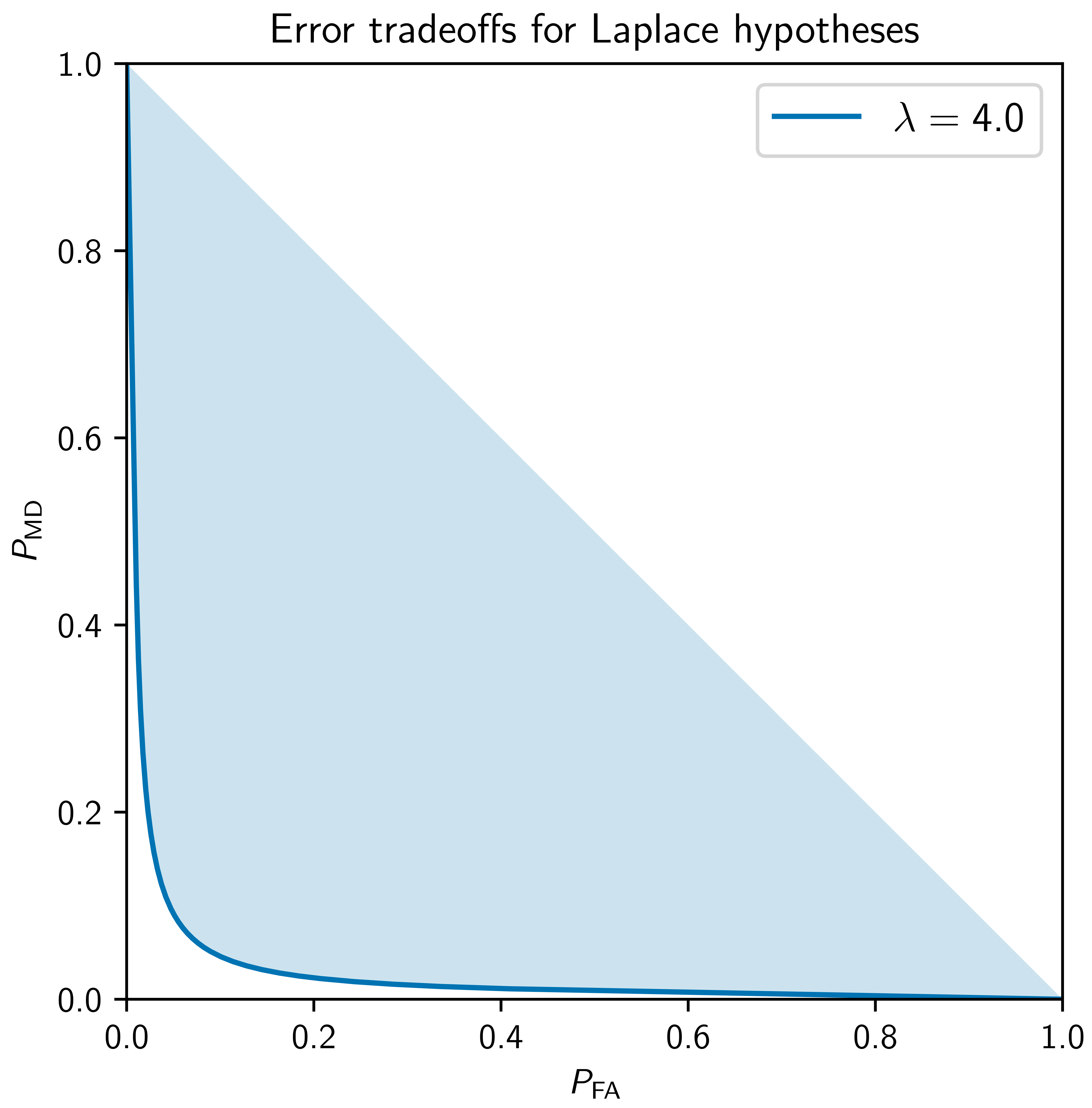

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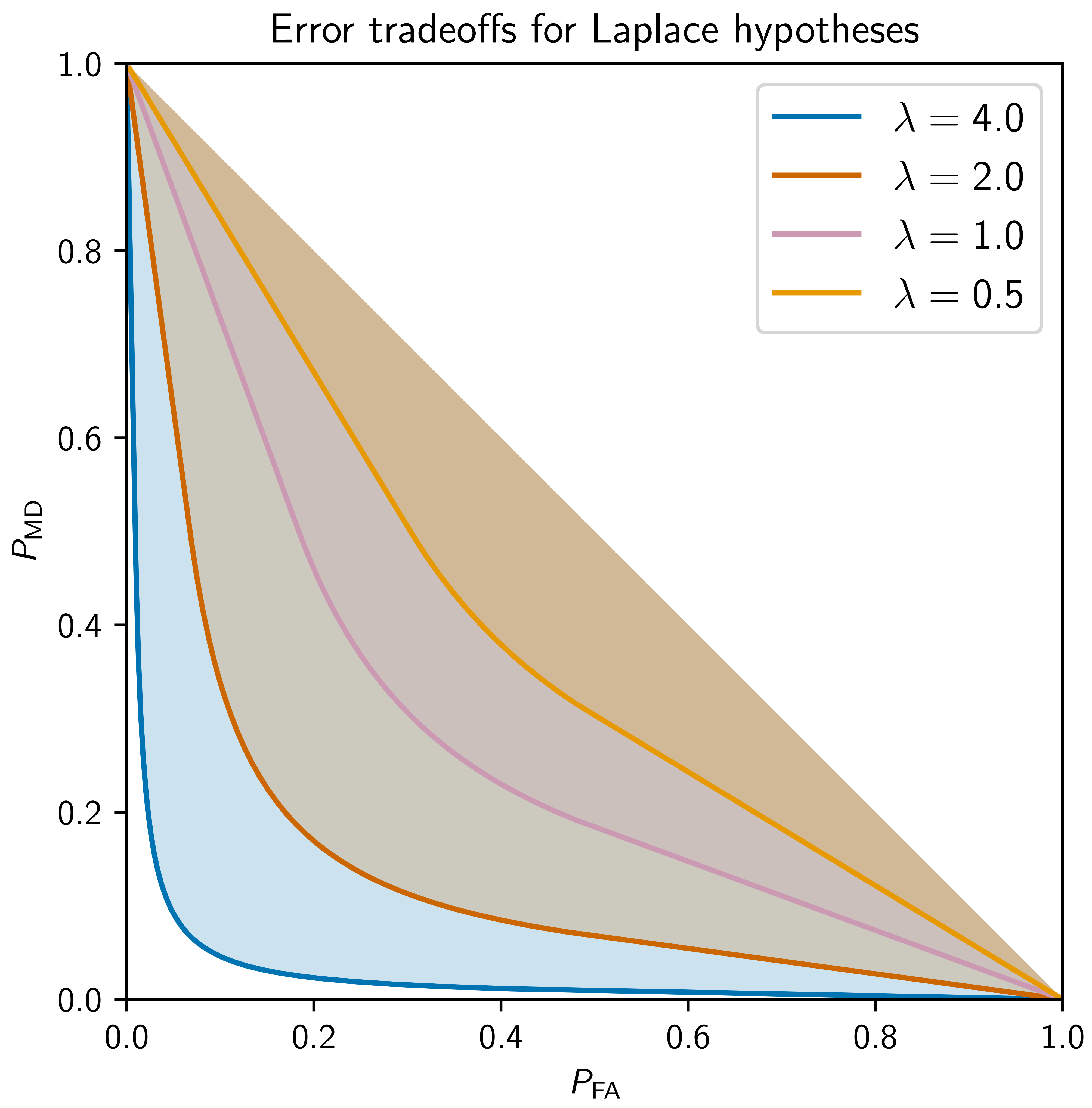

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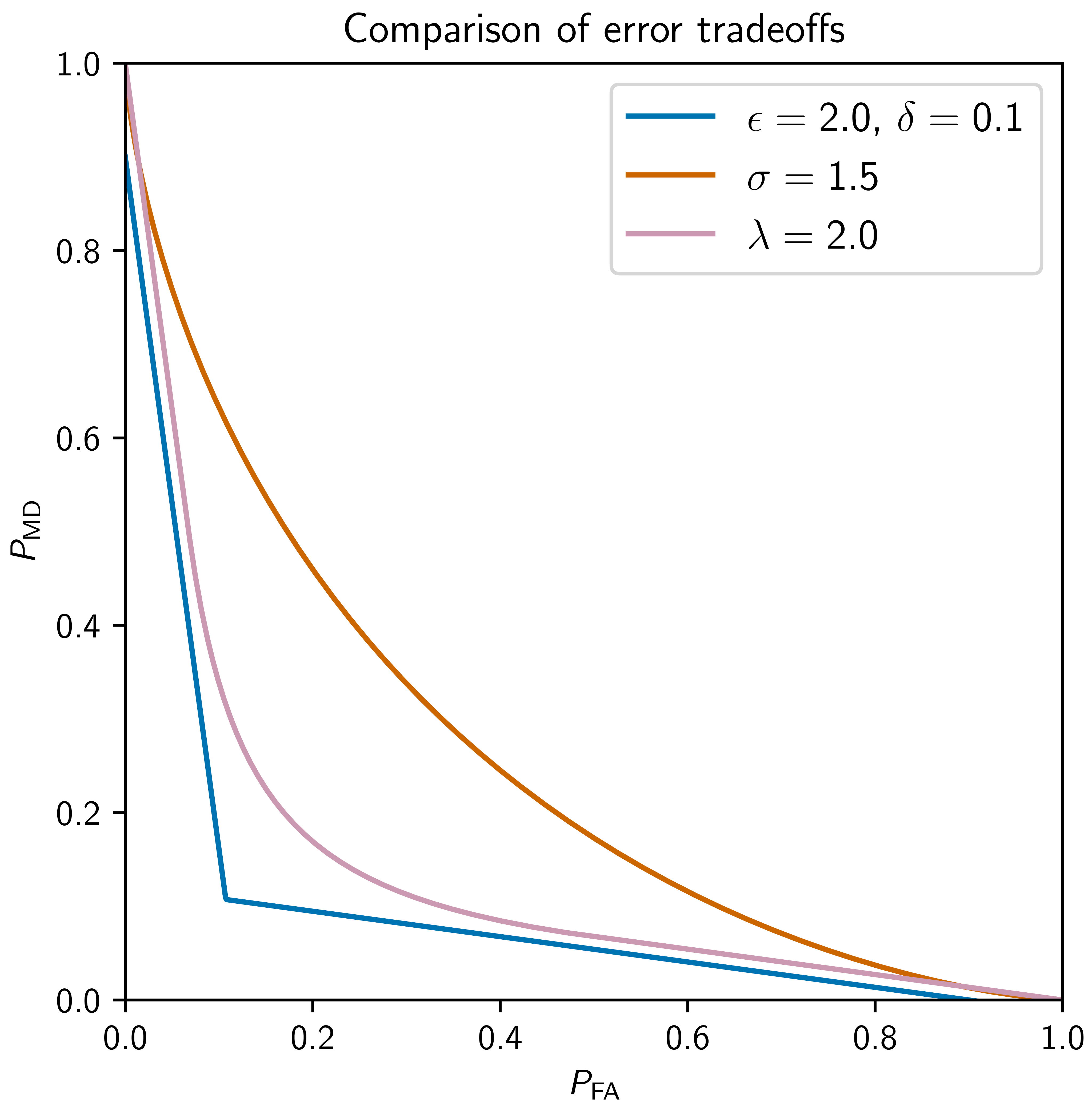

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Wasserman, Zhou (2010) Kairouz, Oh, Vishwanath (2015)

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A **privacy guarantee** is made by the tradeoff between probabilities of **false alarm** (Type I error/false positive) and **missed detection** (Type 2 error/false negative).

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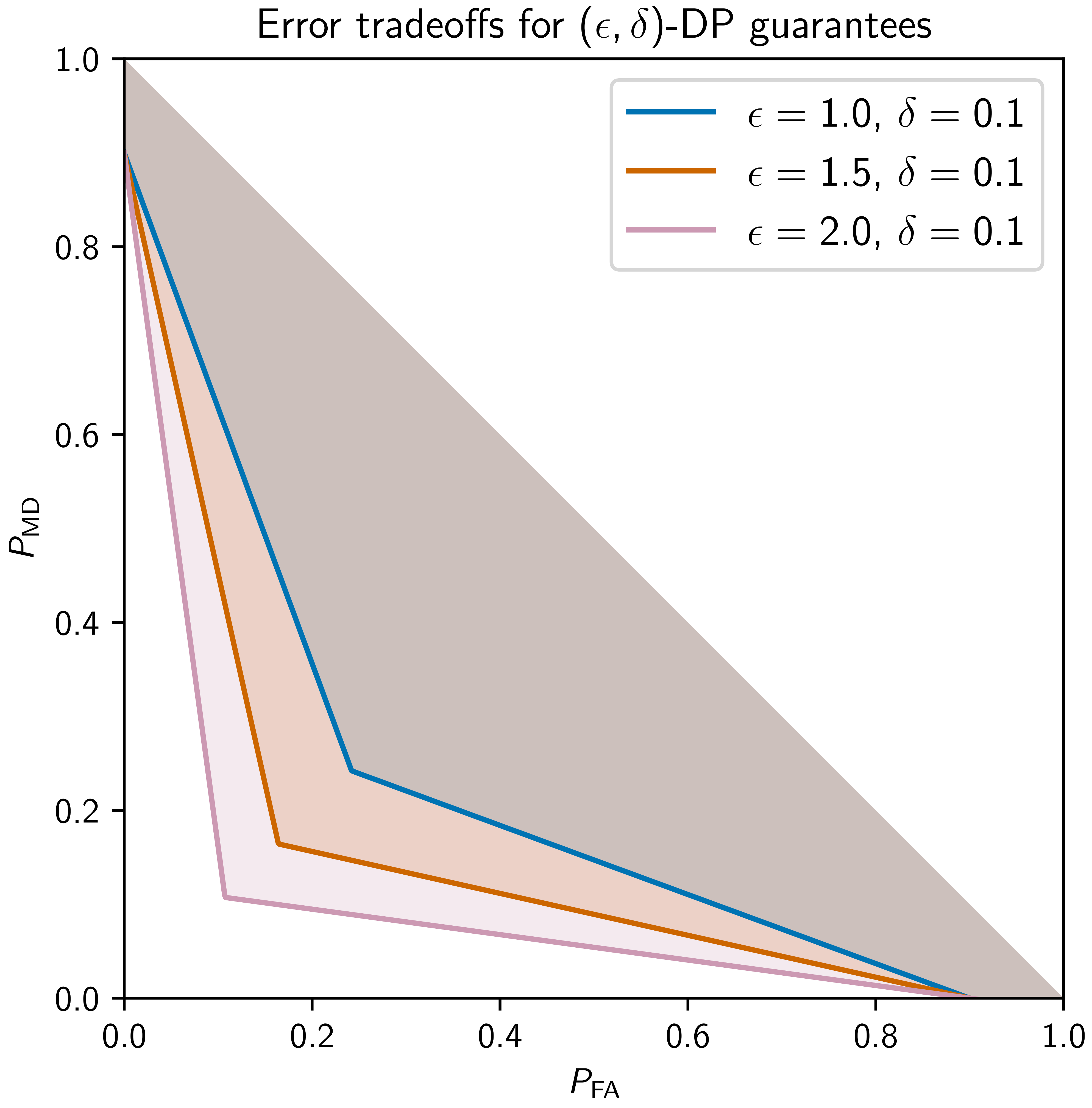

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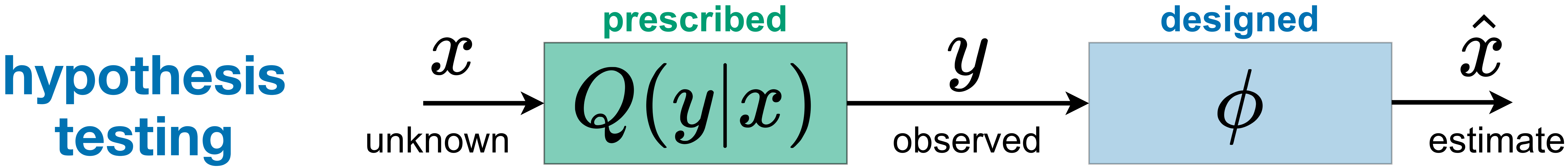

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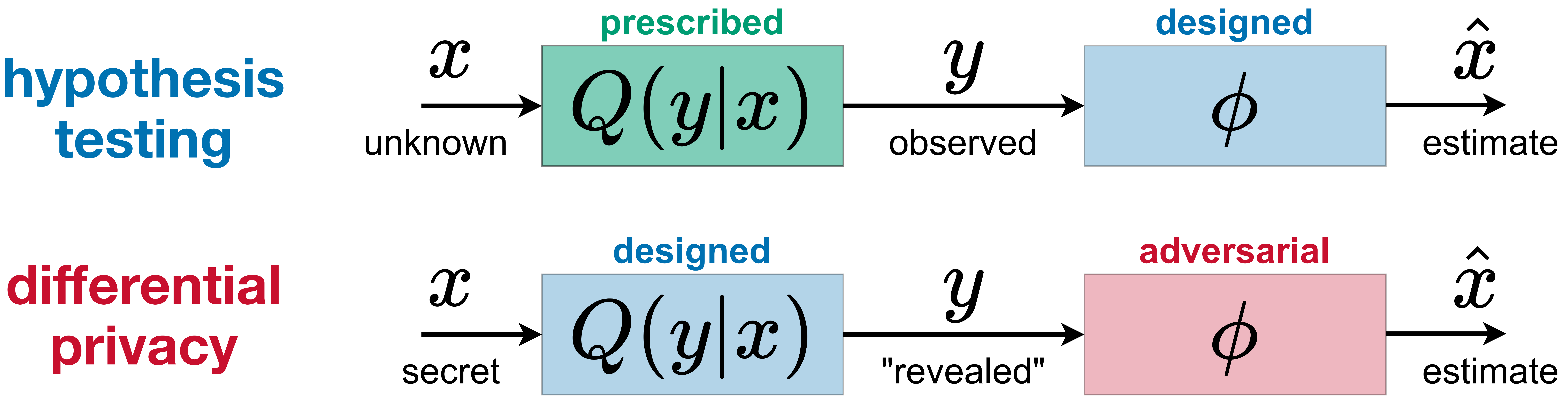

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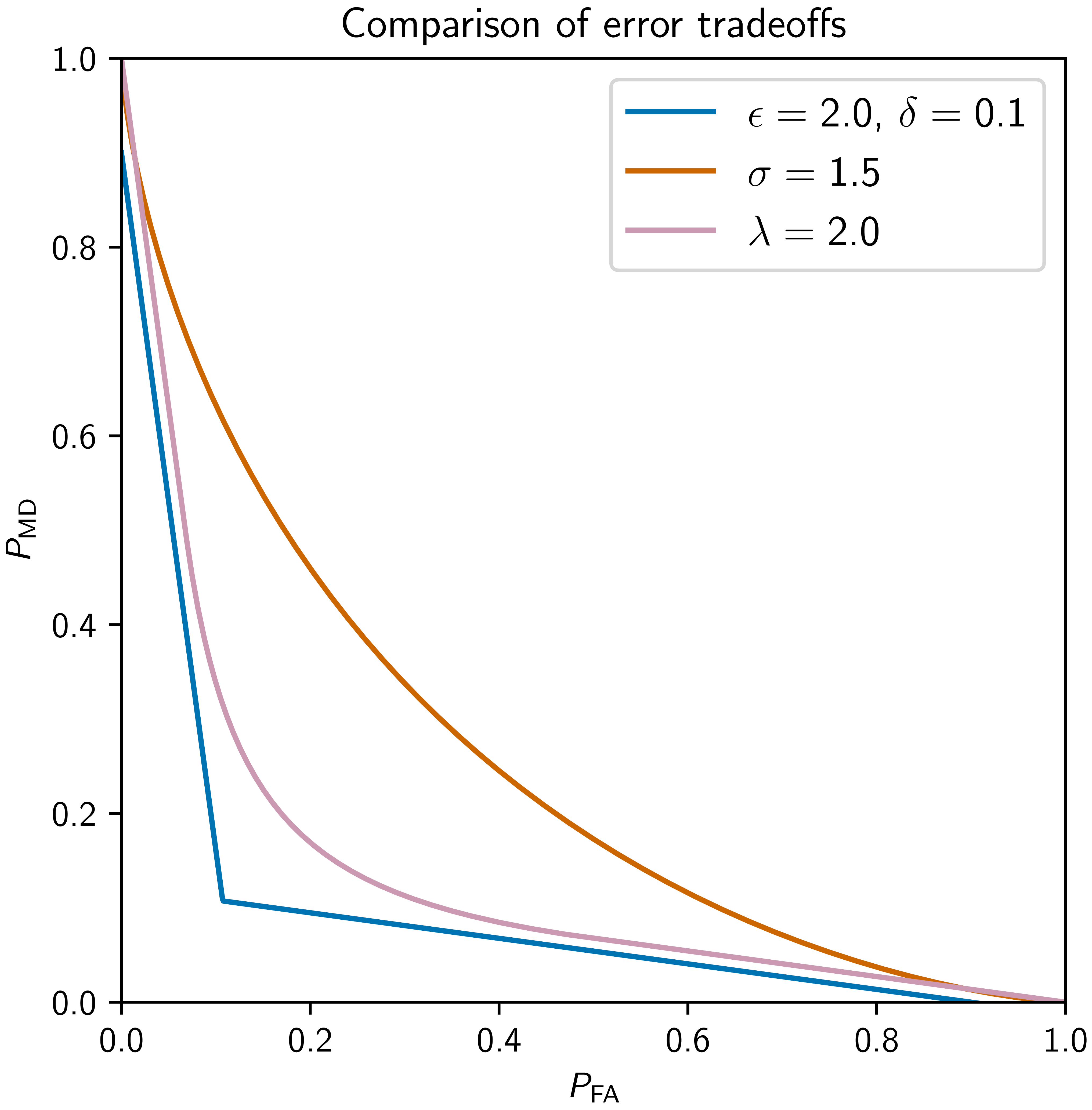

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Vista 2

differential privacy the normal way

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Protecting many single bits simultaneously

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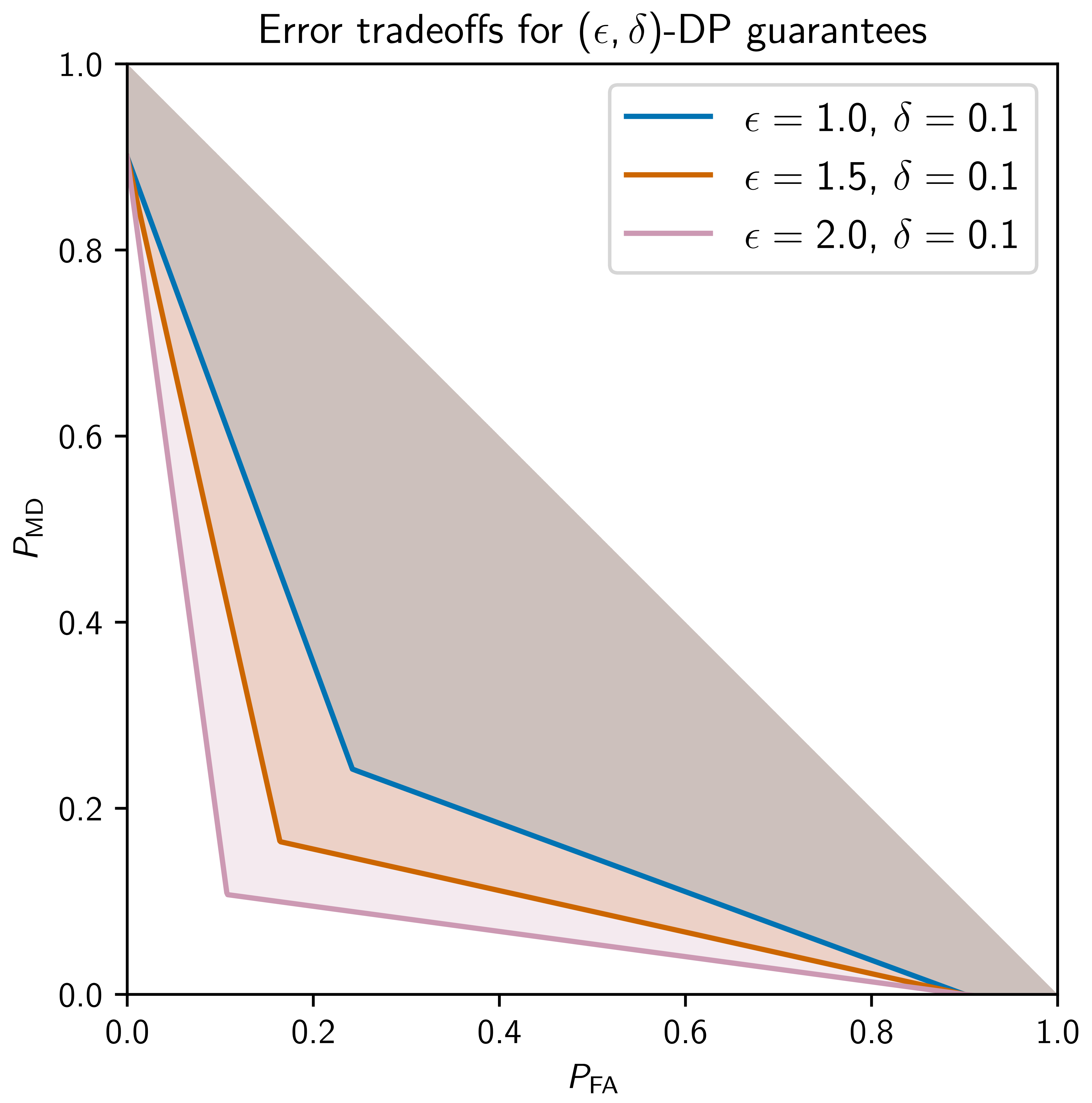

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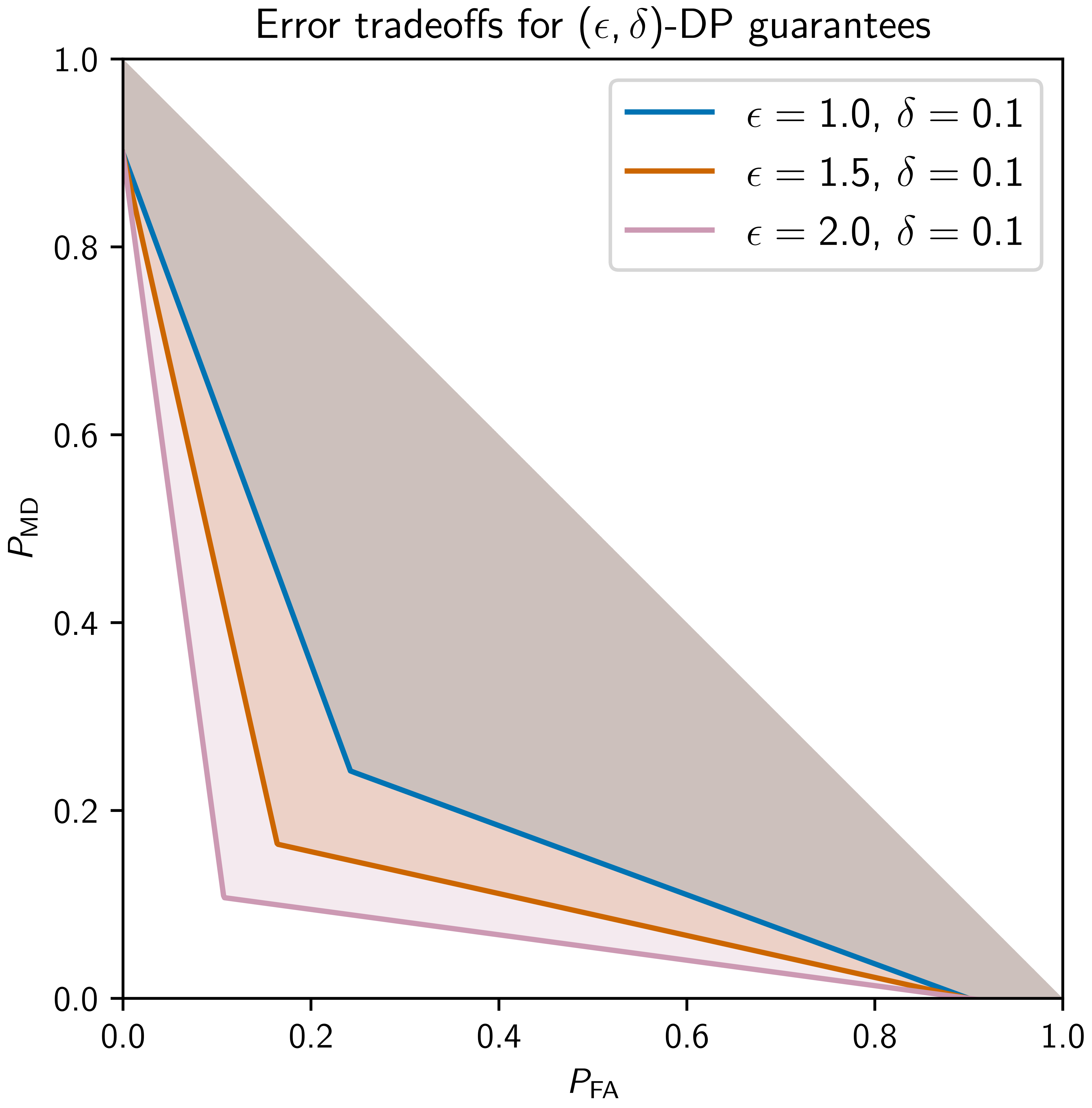

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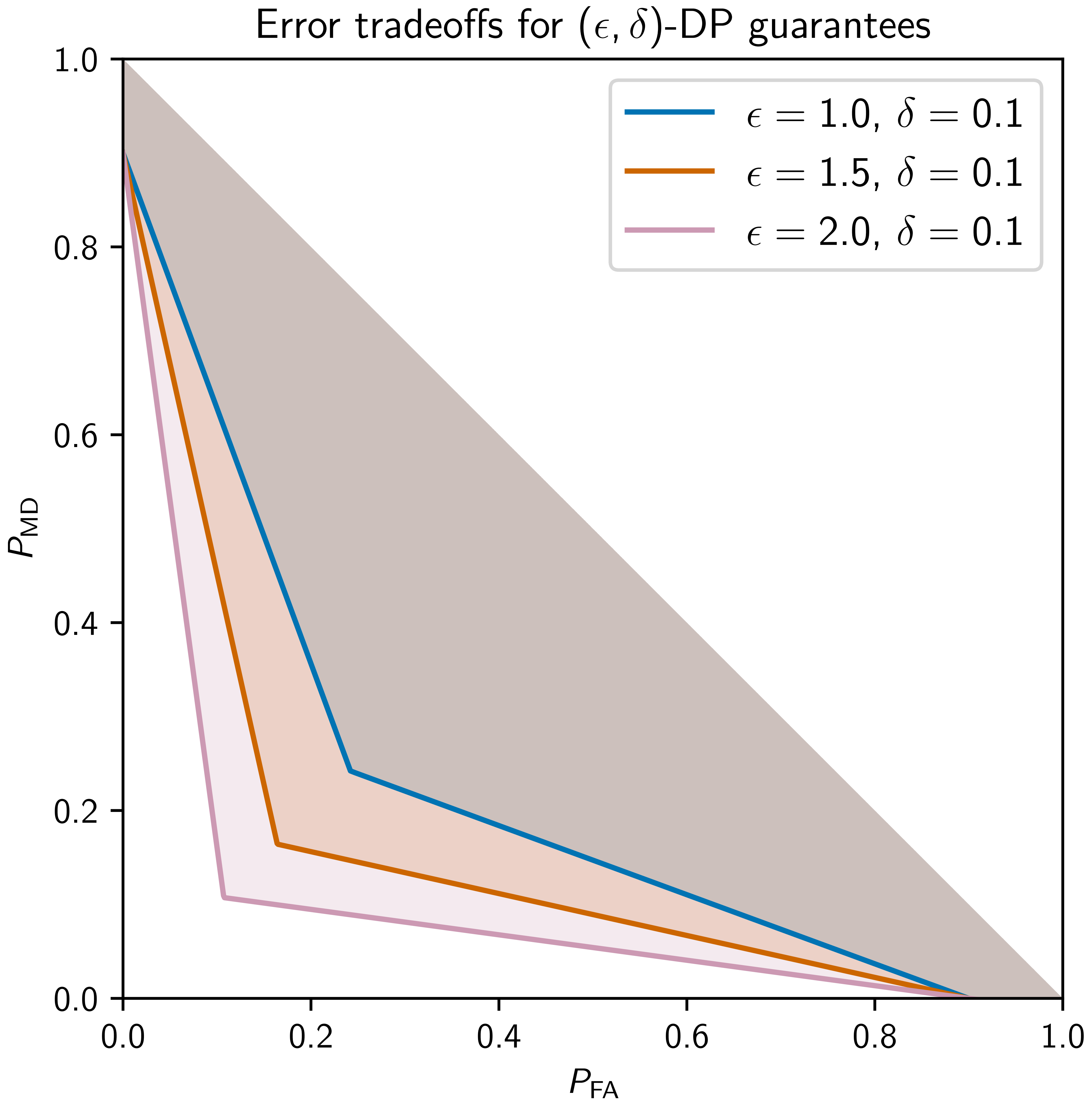

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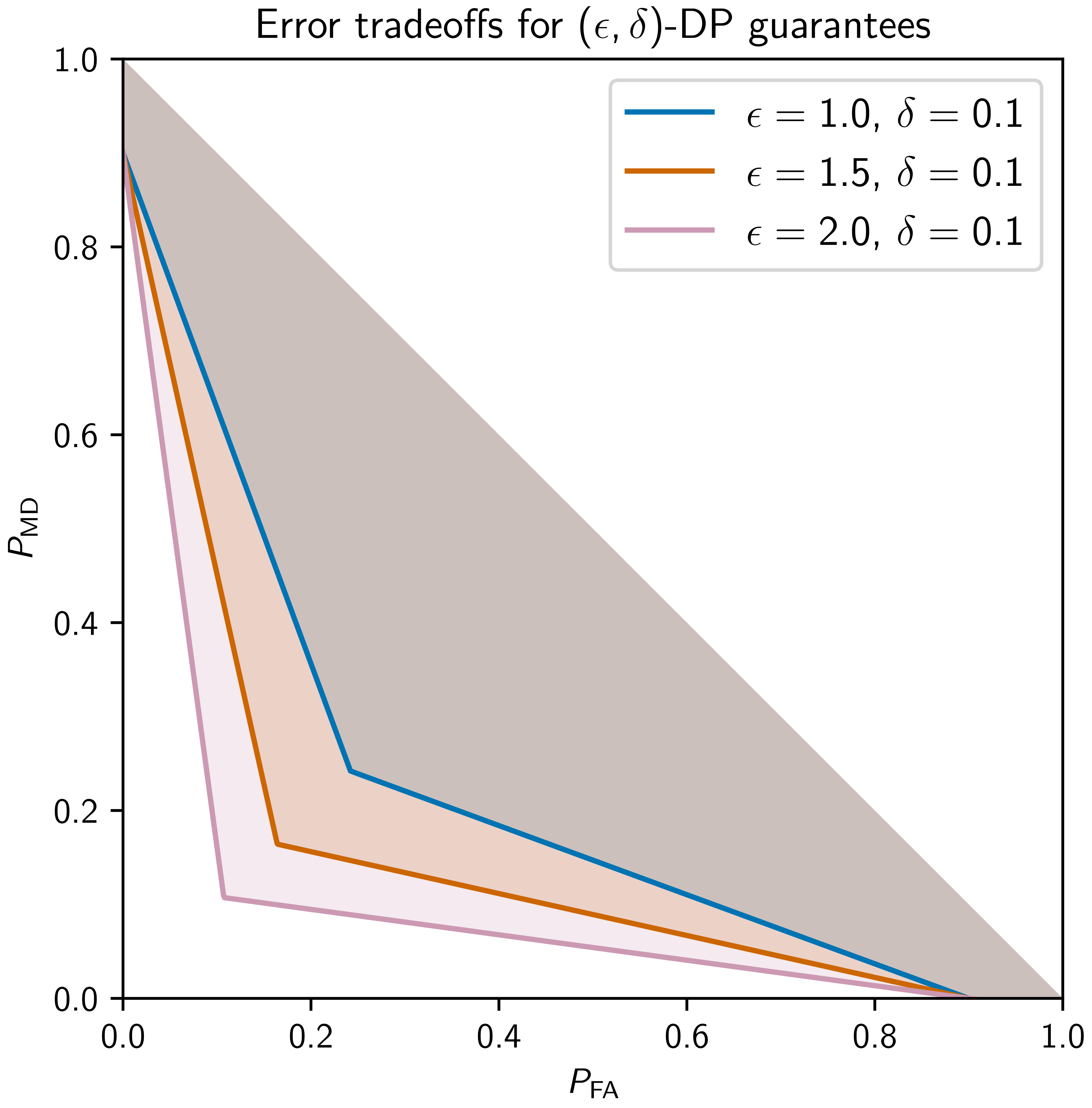

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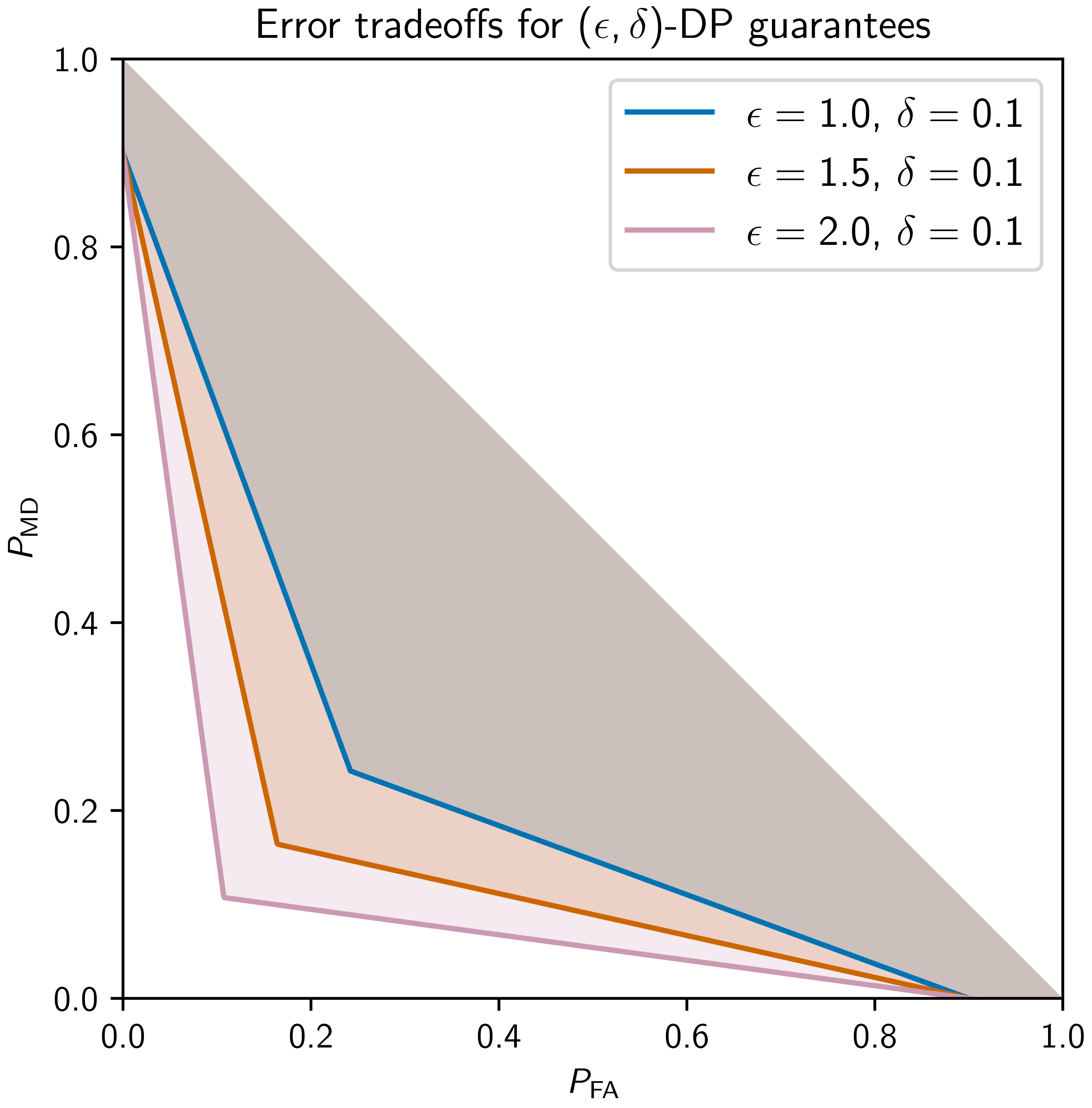

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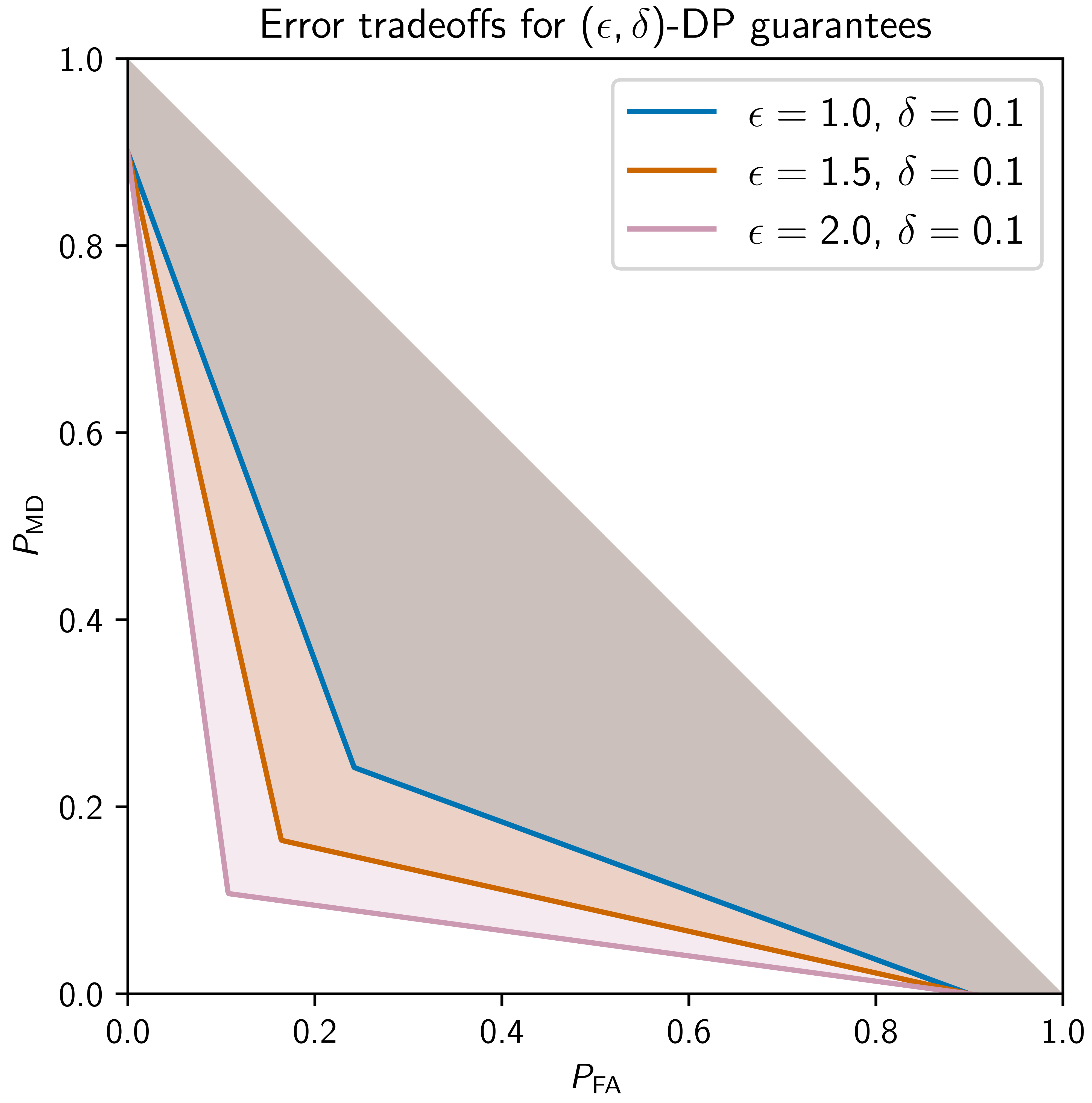

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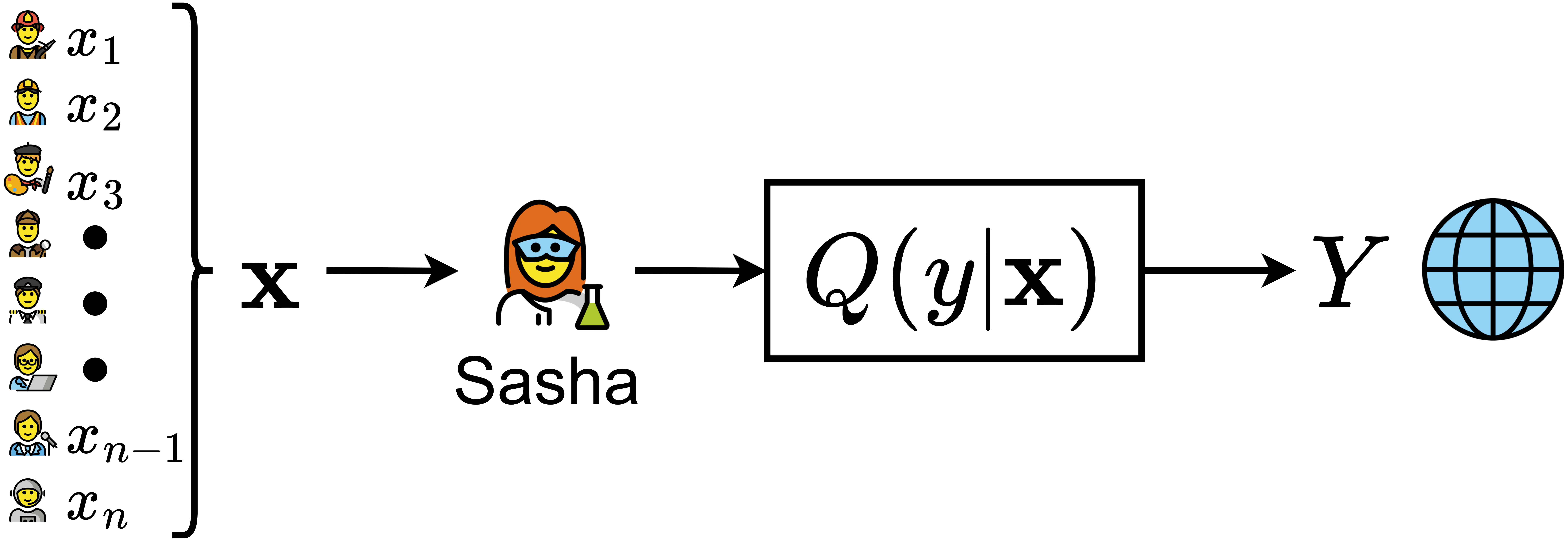
When can we do this? When **neighboring data sets make similar output distributions**.

In a snapshot

Replacing a single bit with a database



The hypothesis testing in DP

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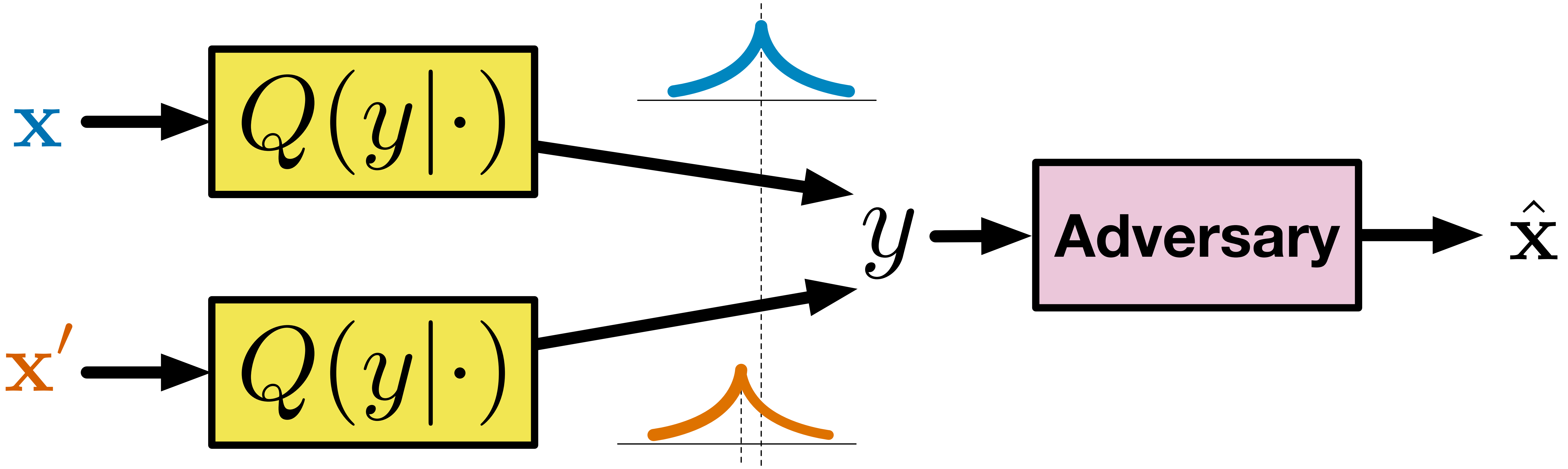

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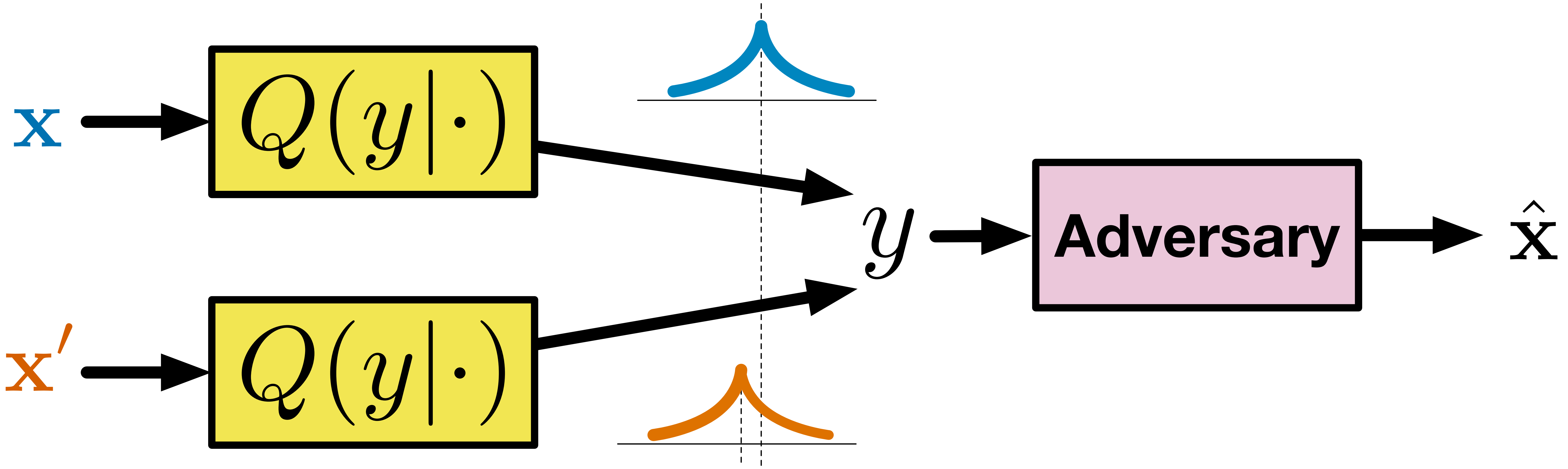

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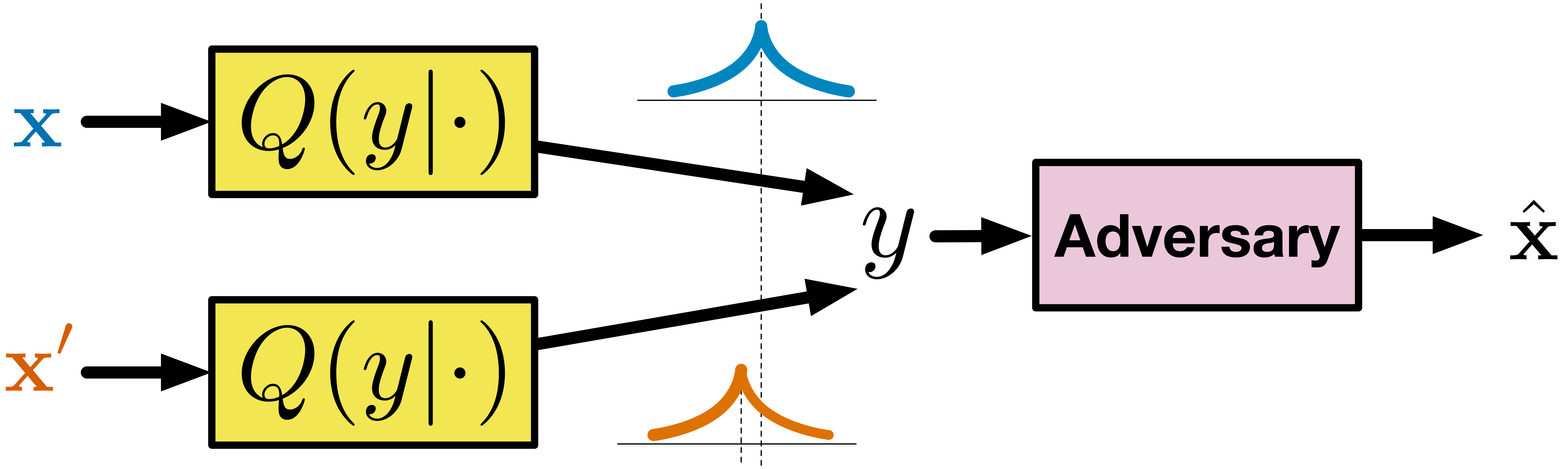

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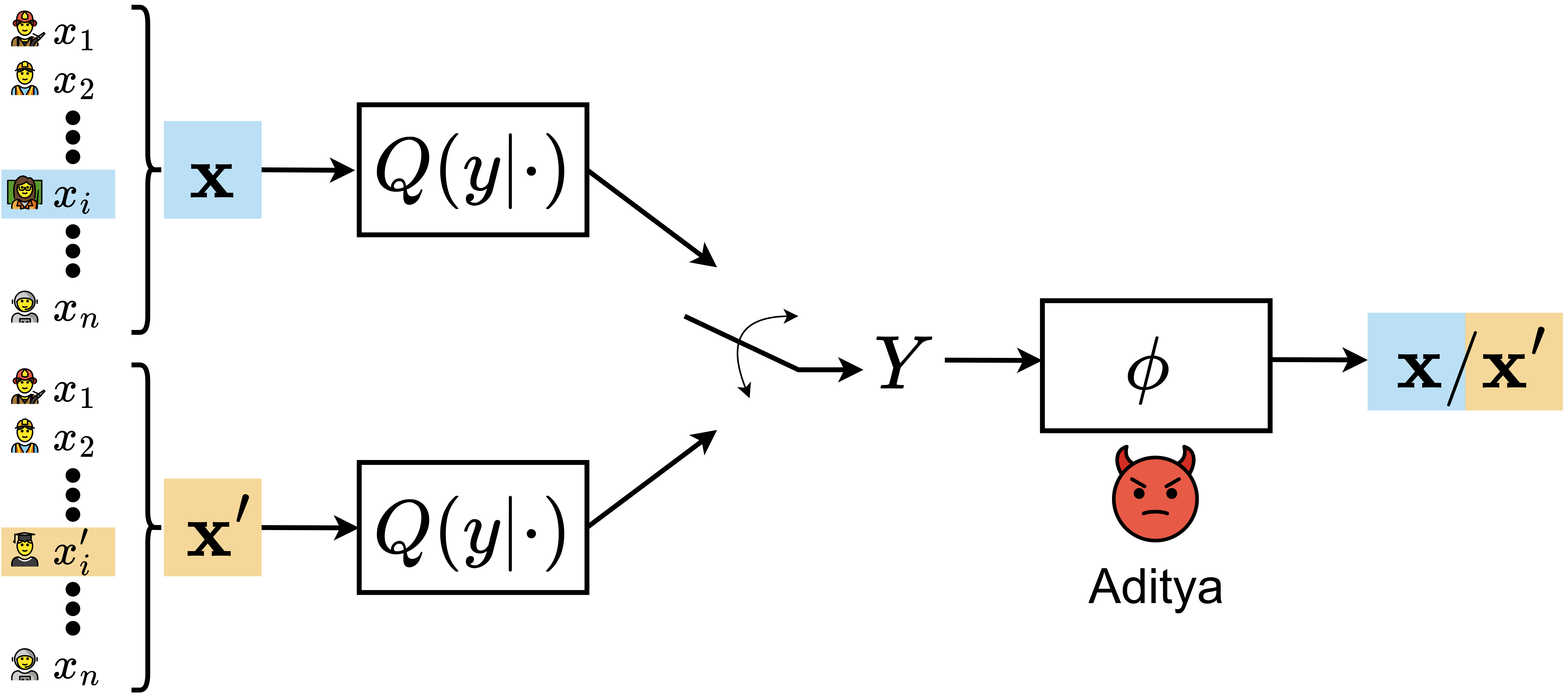
[Dwork-Kenthapadi-McSherry-Mironov-Naor 2006]

[Wasserman-Zhou 2010]

Neighboring datasets in a picture

The adversary's hypothesis test



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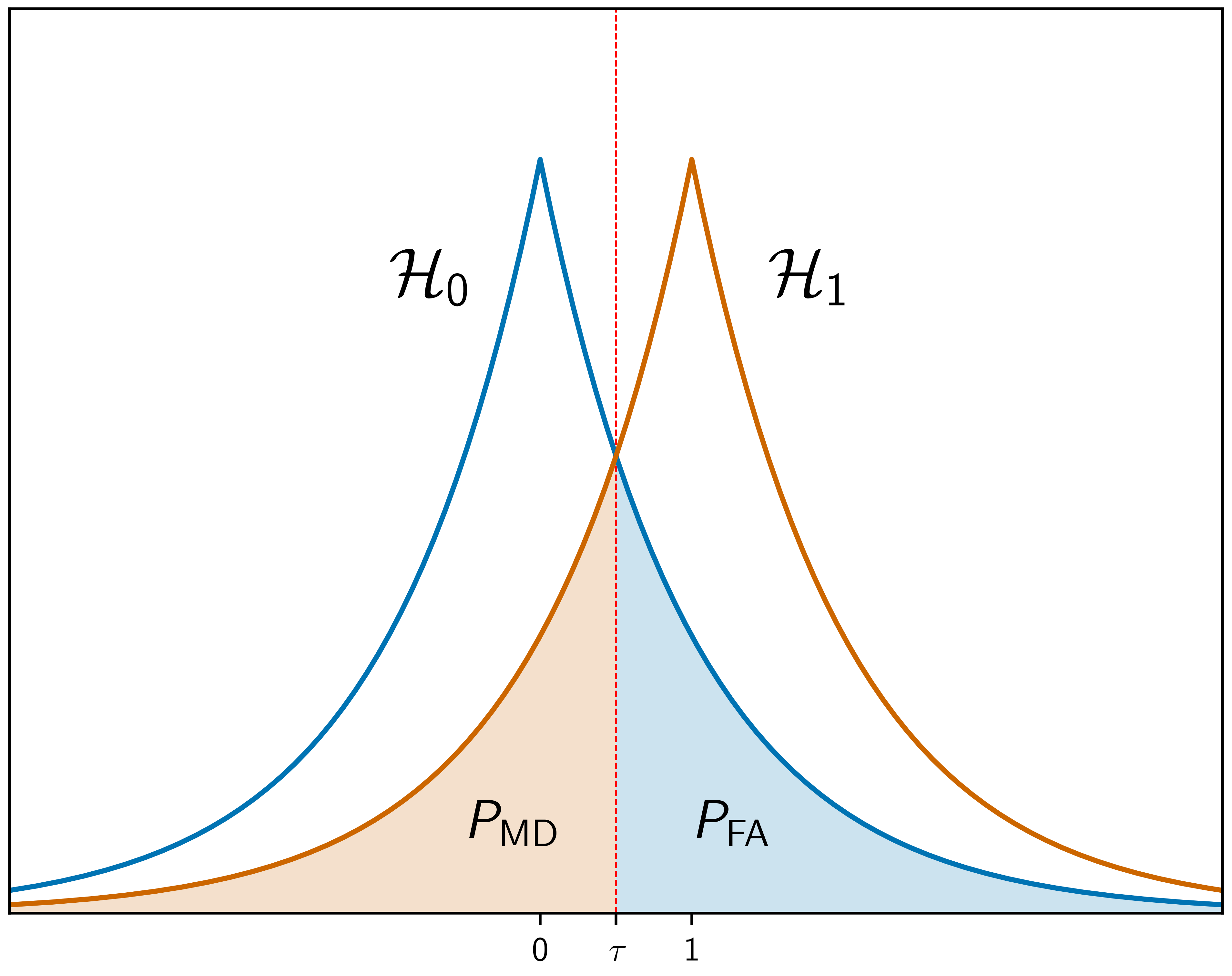
Sensitivity of scalar functions

Understanding the distance between hypotheses



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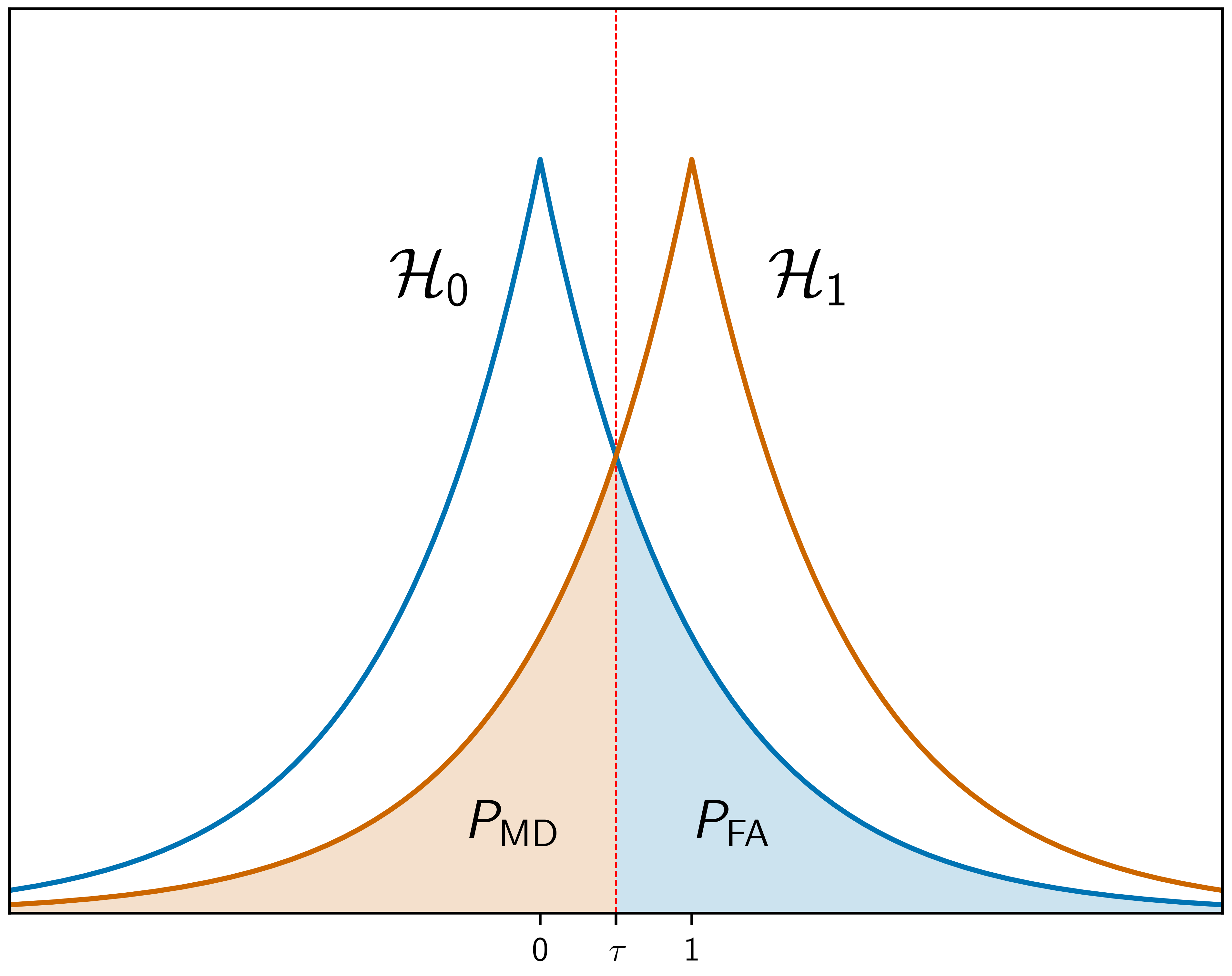

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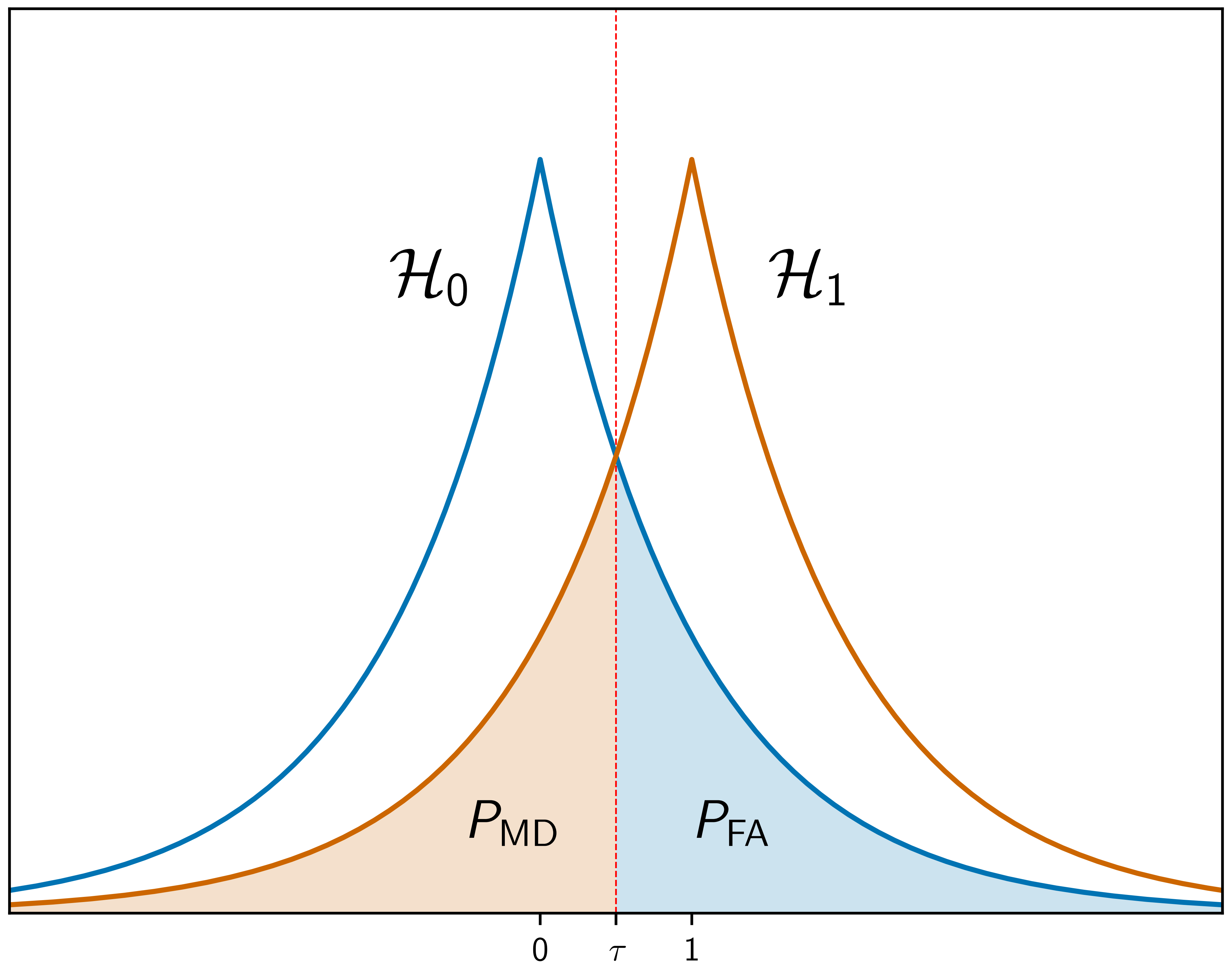

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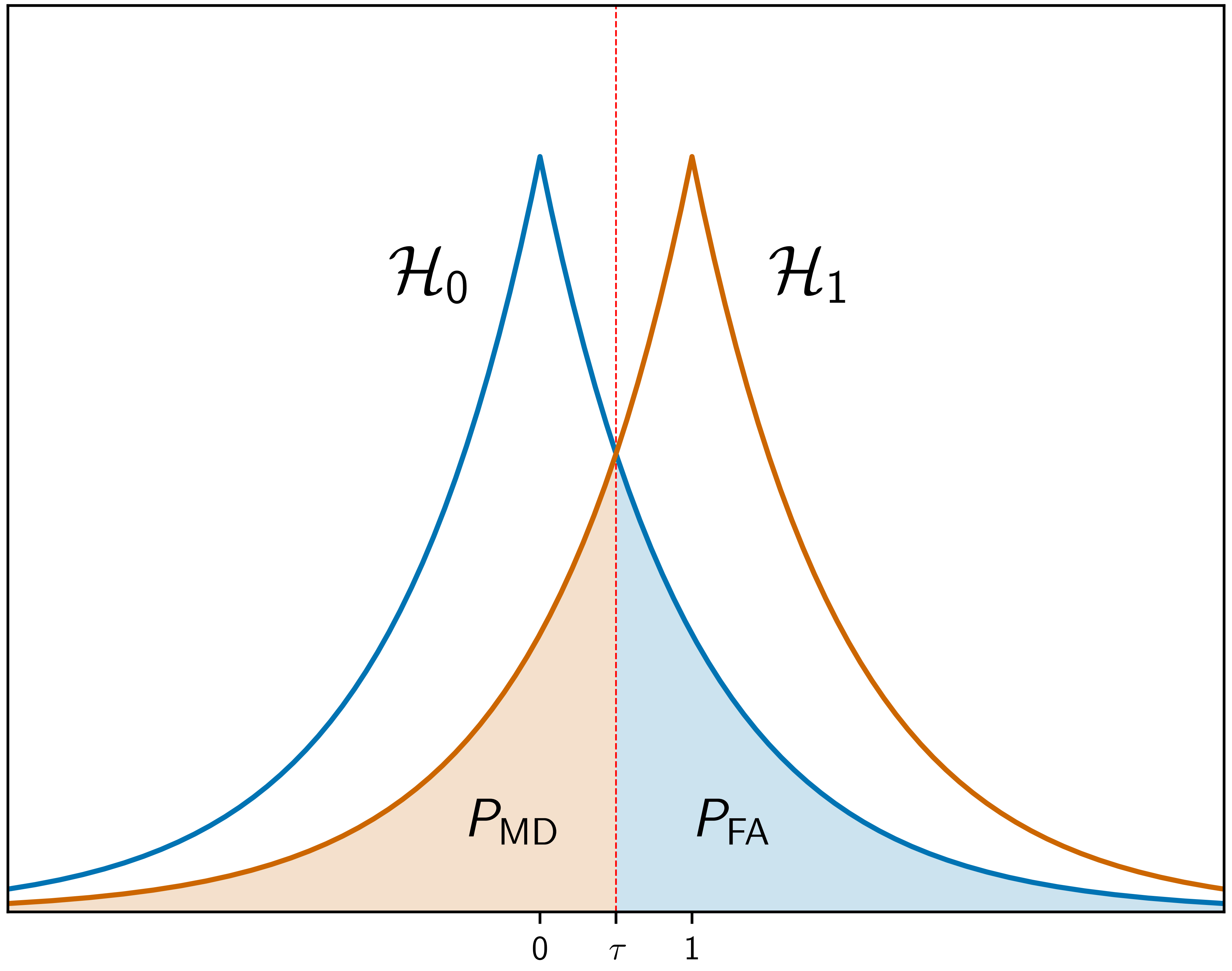

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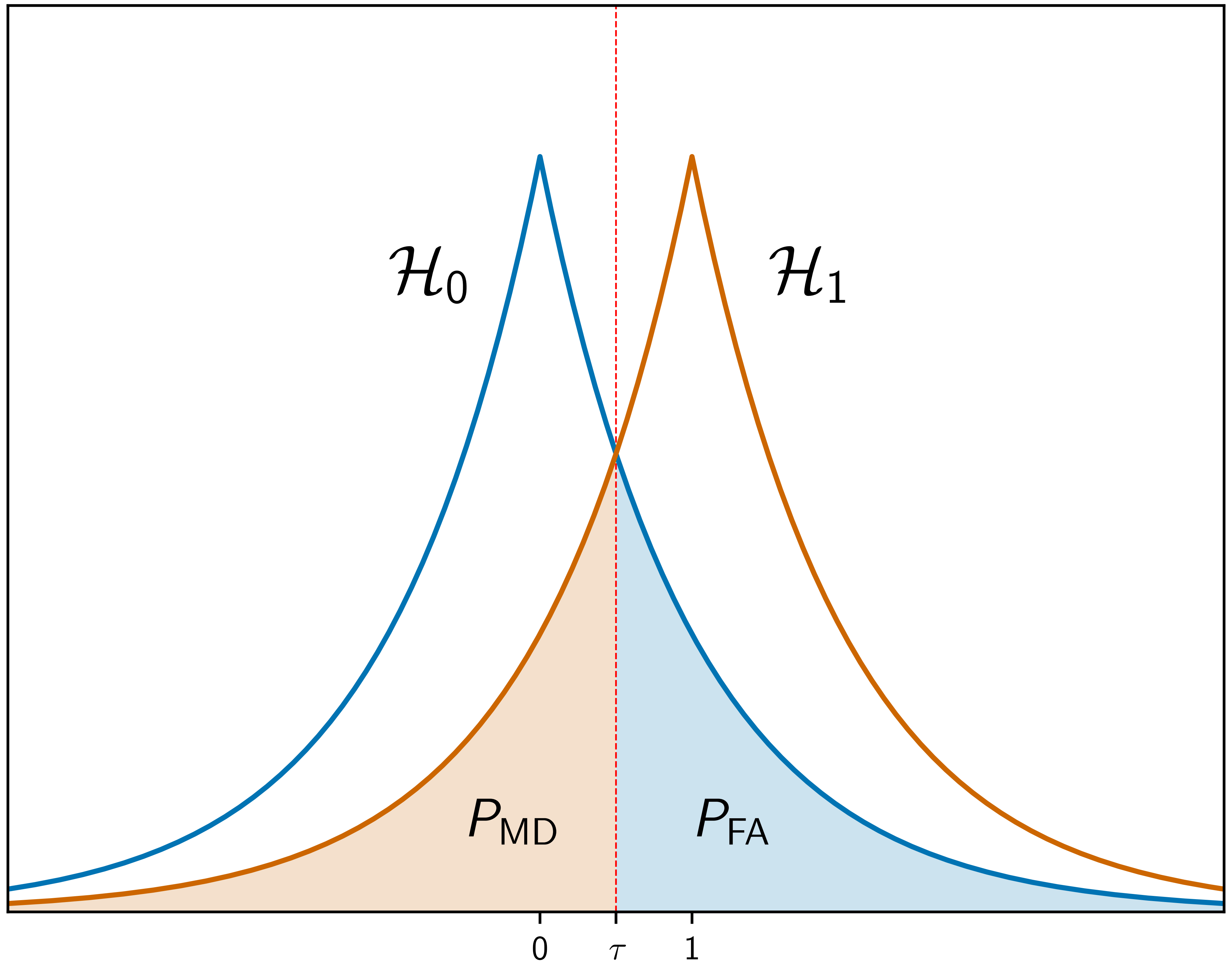

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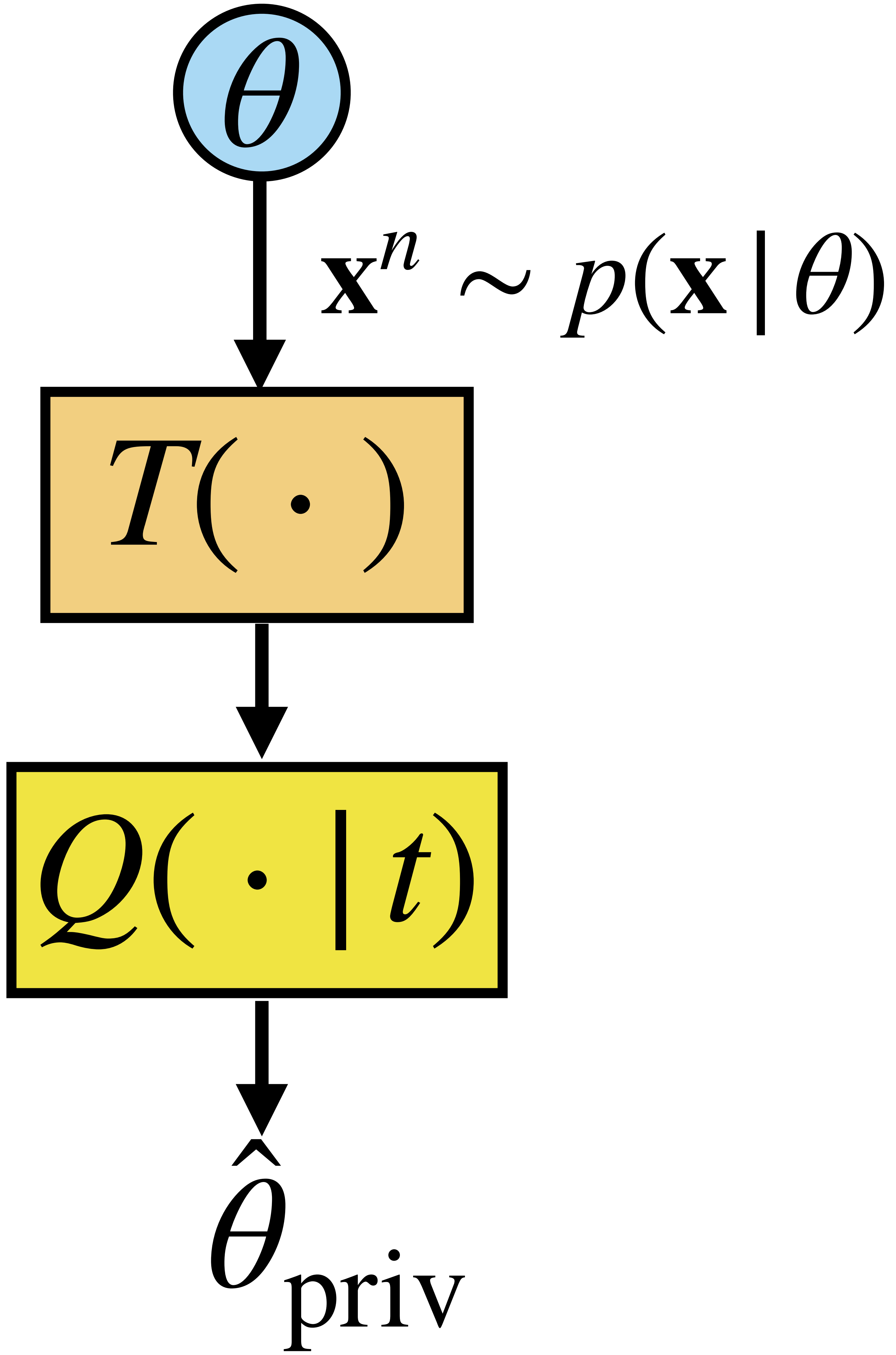
This is what people call the **privacy-utility tradeoff**.

Point estimation with differential privacy

Adding noise to sufficient statistics



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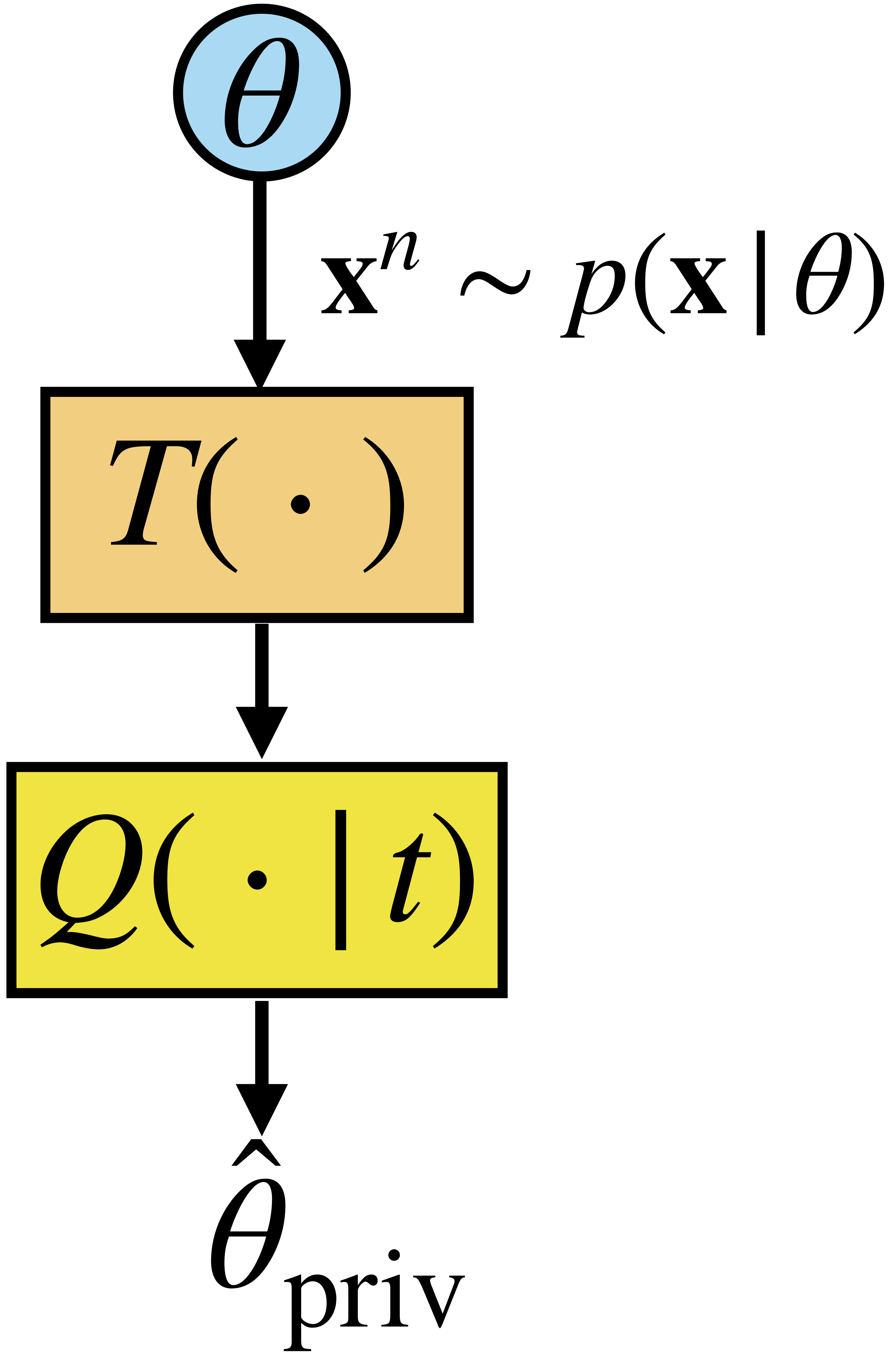

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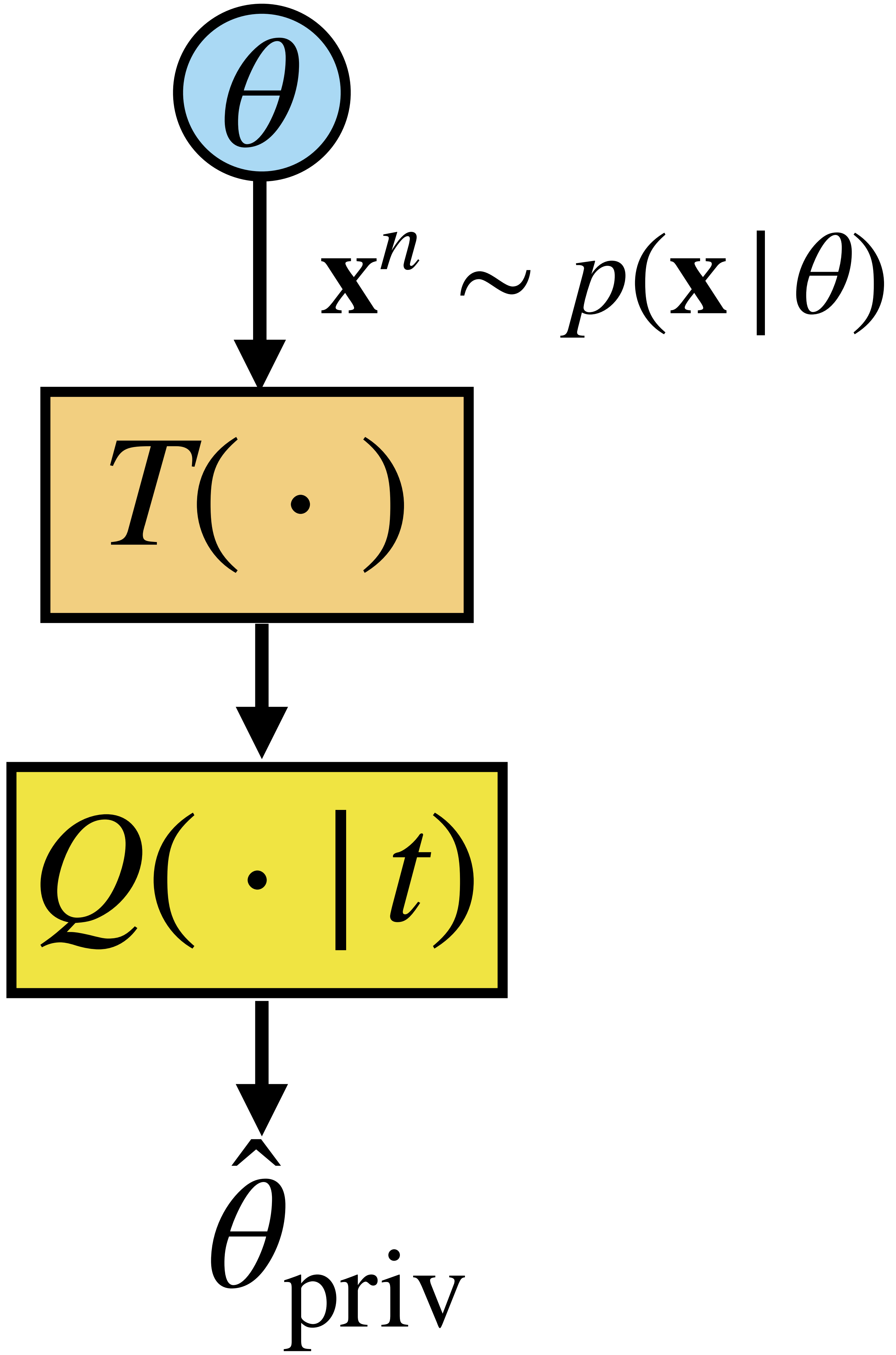

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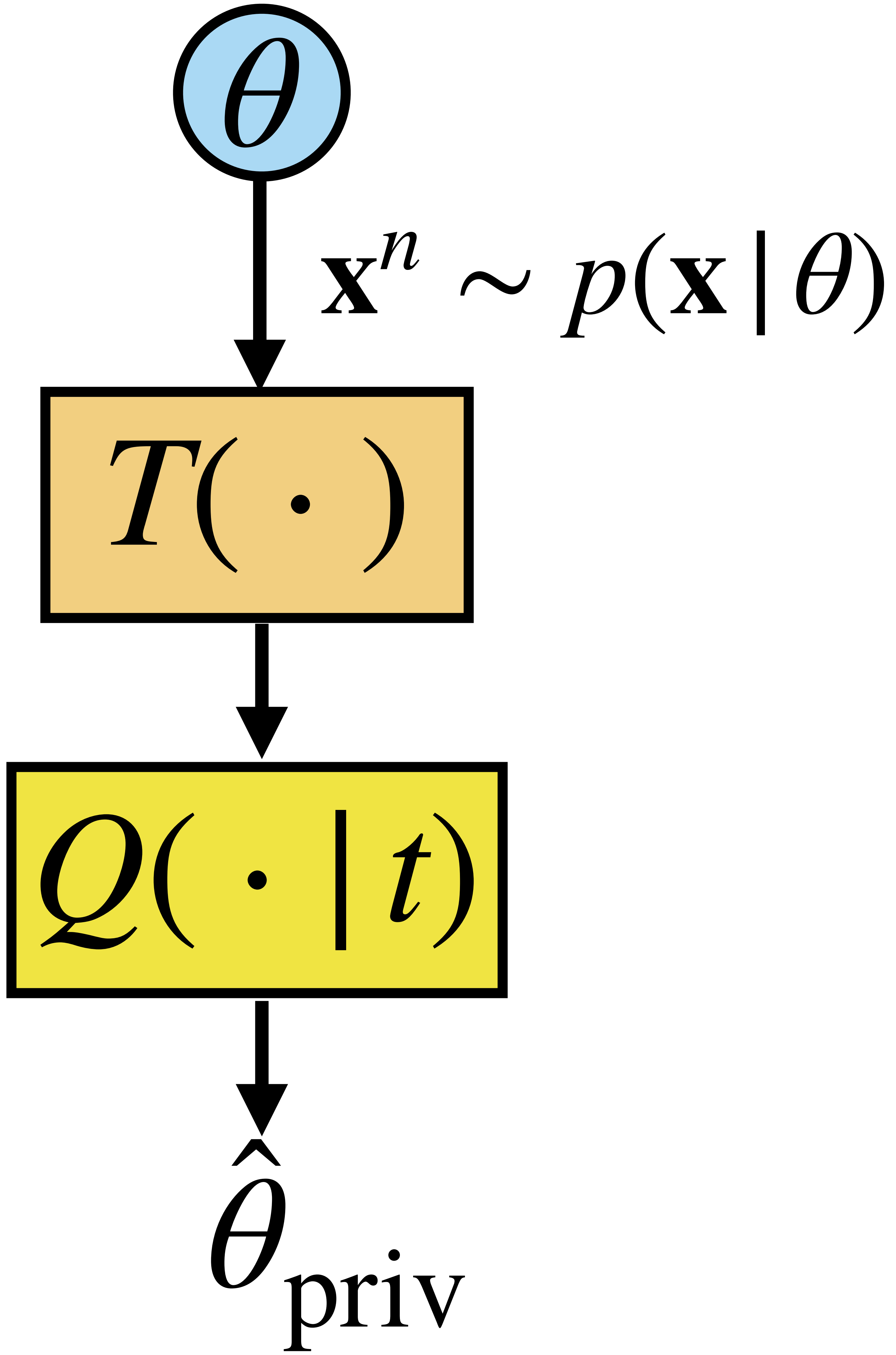

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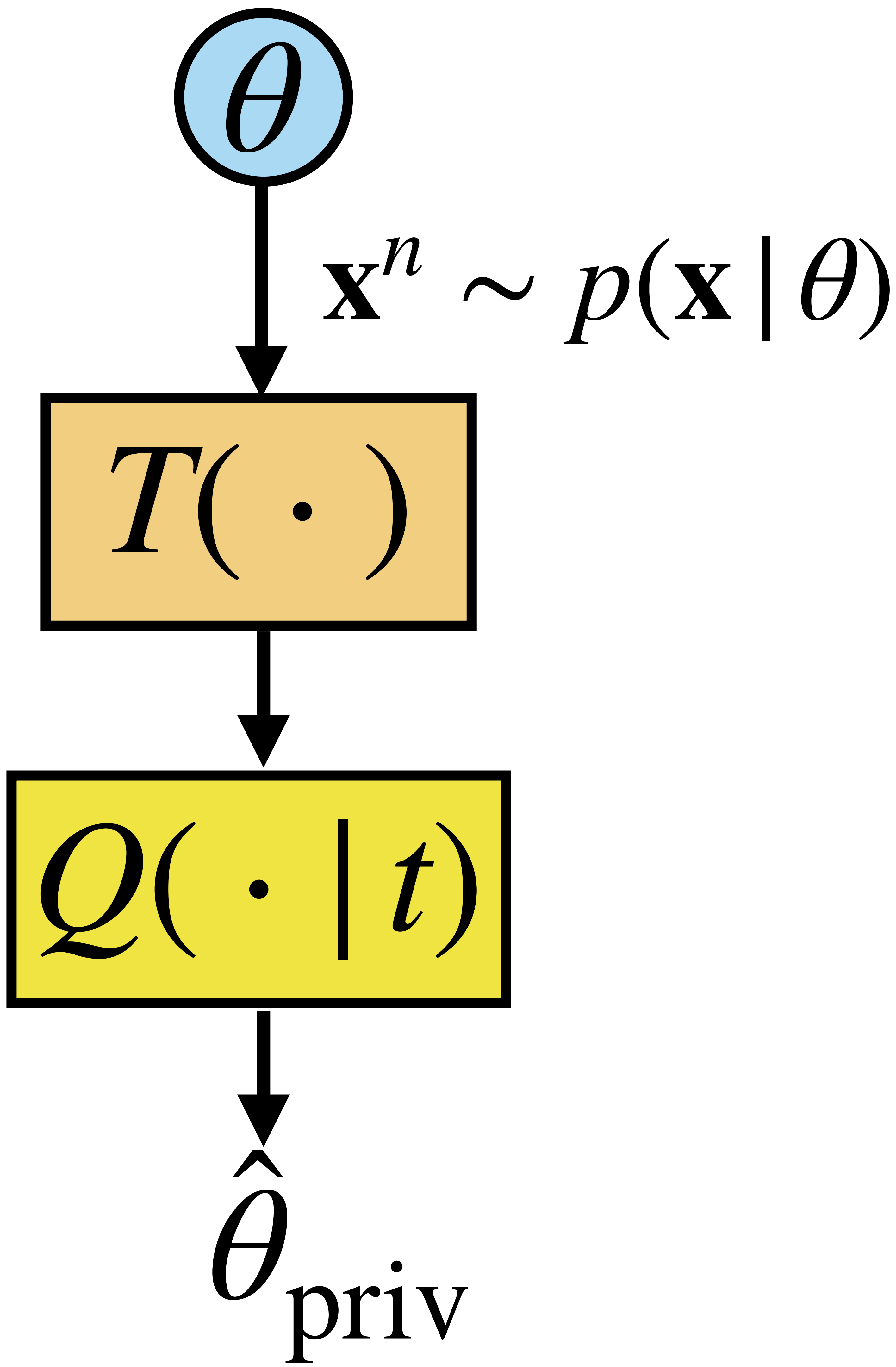

A typical DP approach to statistical estimation (Smith 2009):

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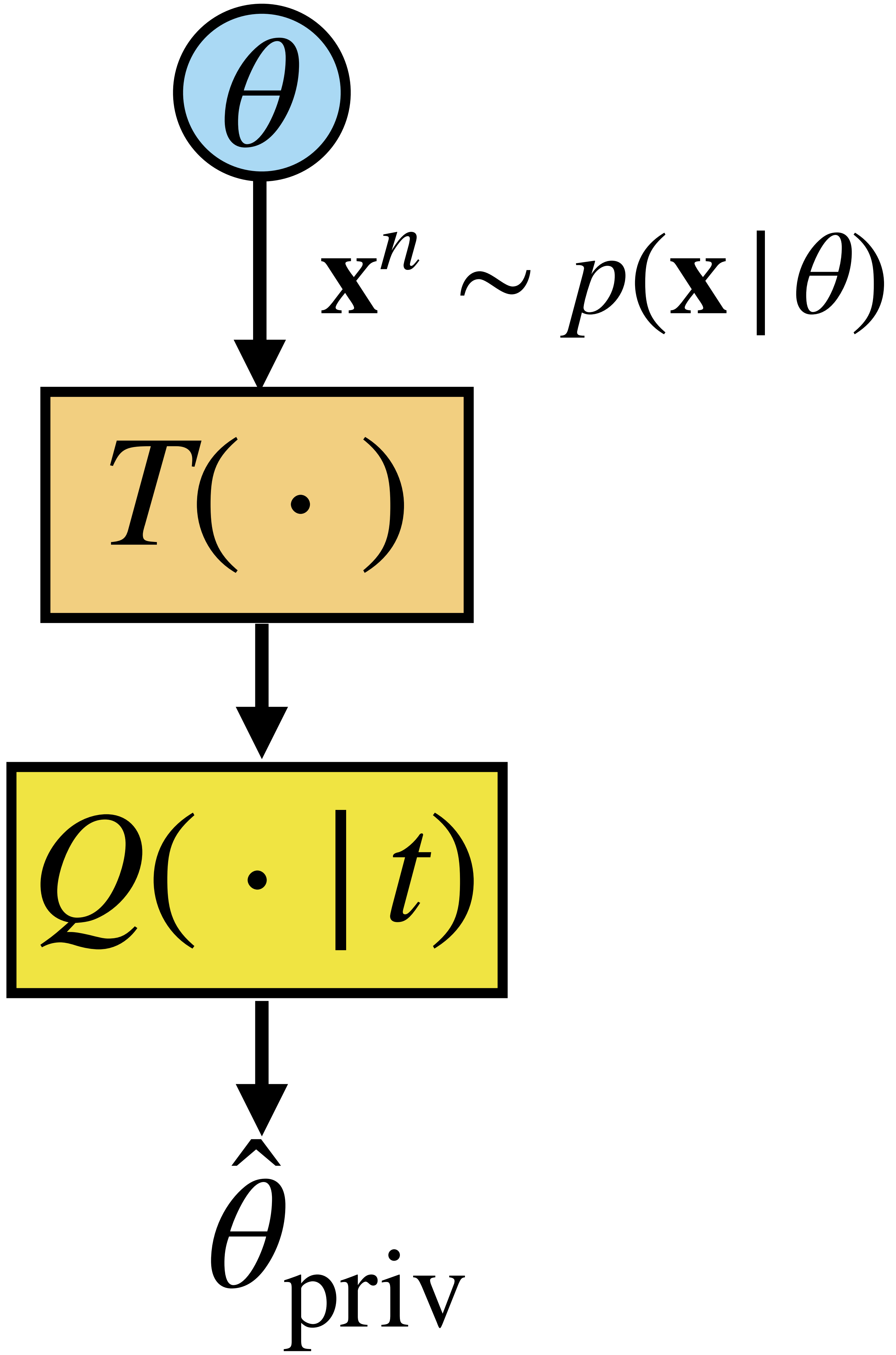

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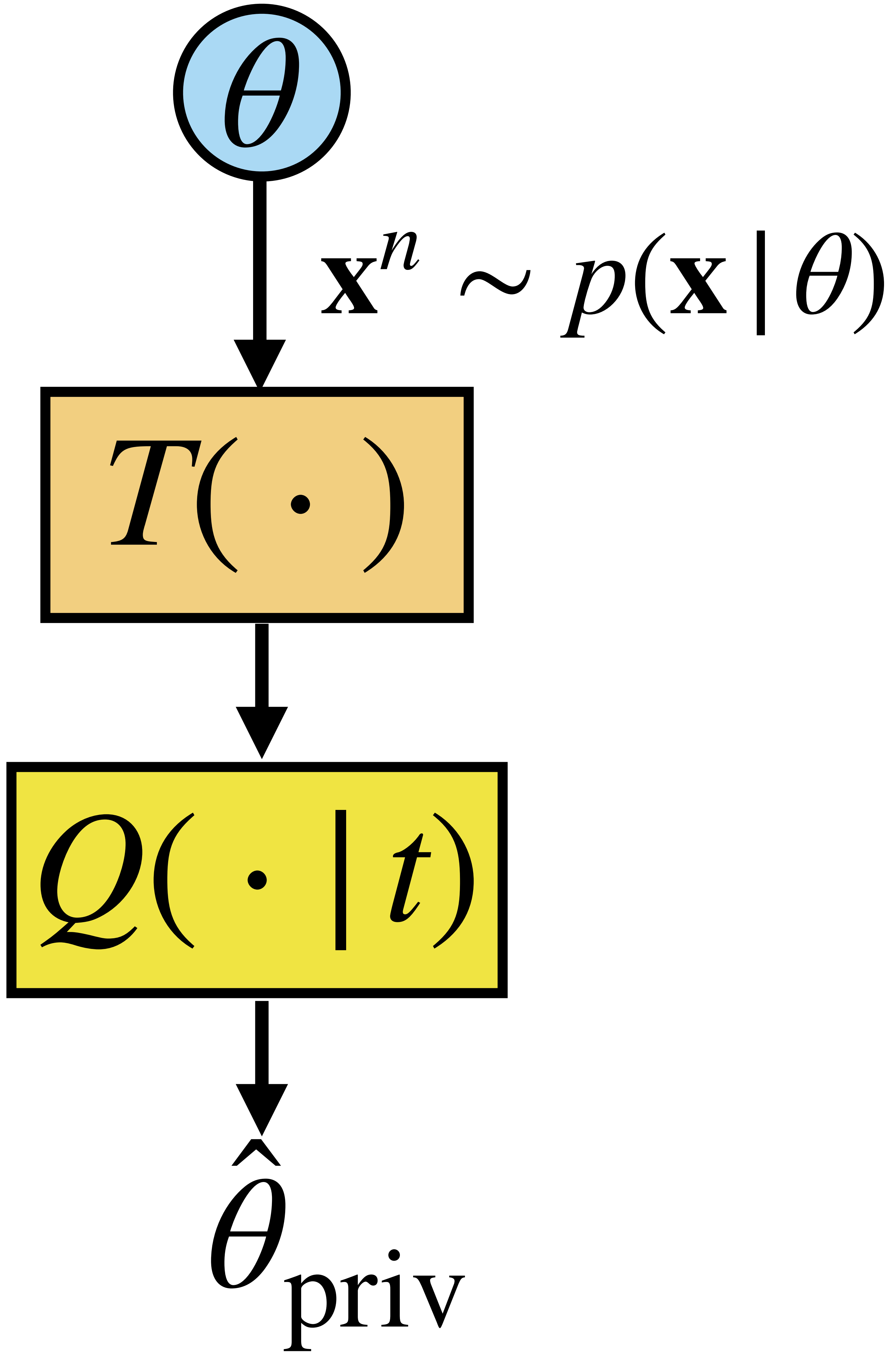

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All we need to know is the sensitivity of $T(\,\cdot\,)$.

What kind of noise should we use?

So many different choices: a non-comprehensive list

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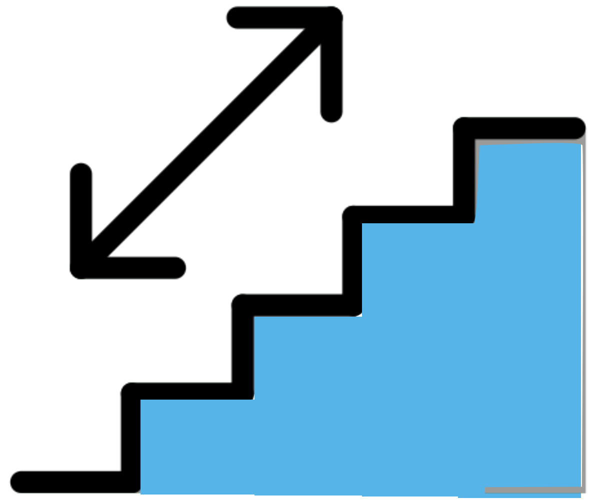

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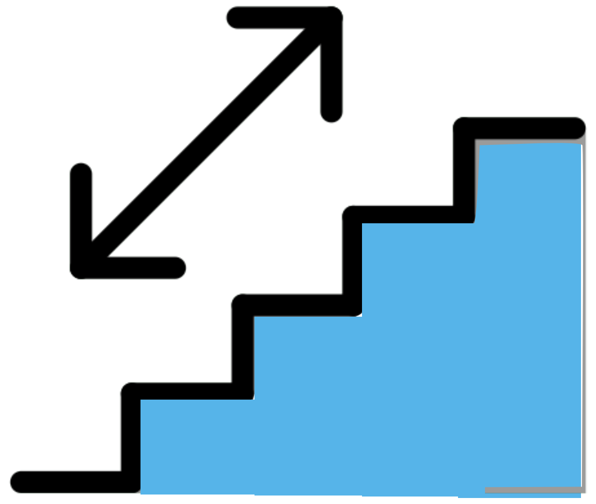

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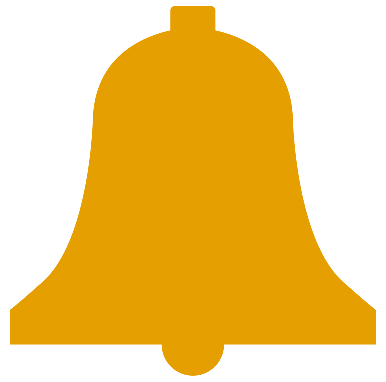

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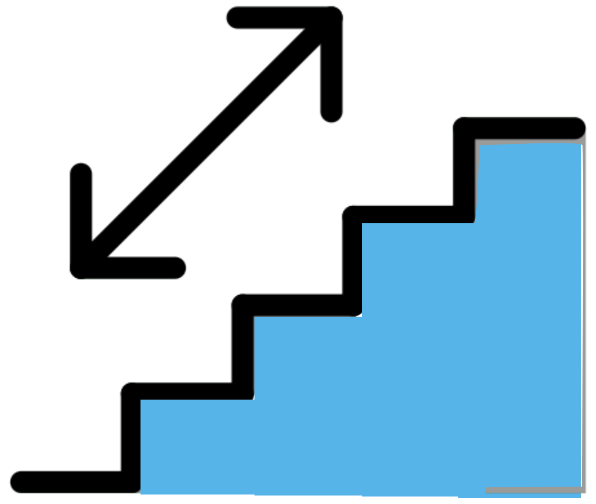

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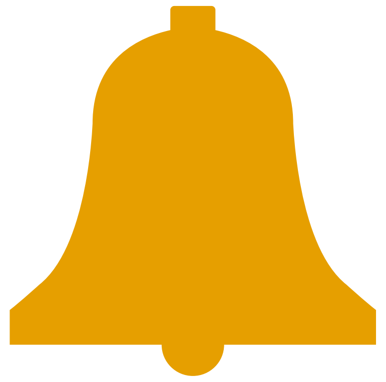

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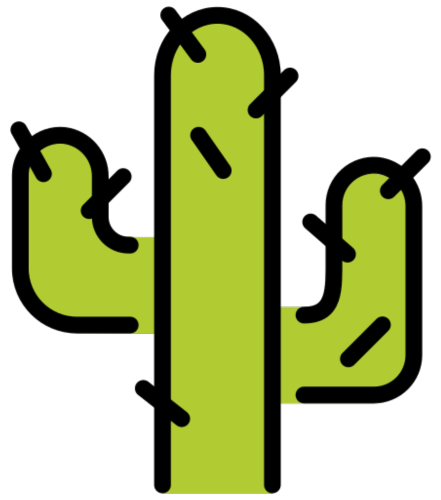

“Other"

Geng, Ding, Guo, Kumar (2019/2020)

Dong, Su, Zhang (2021)

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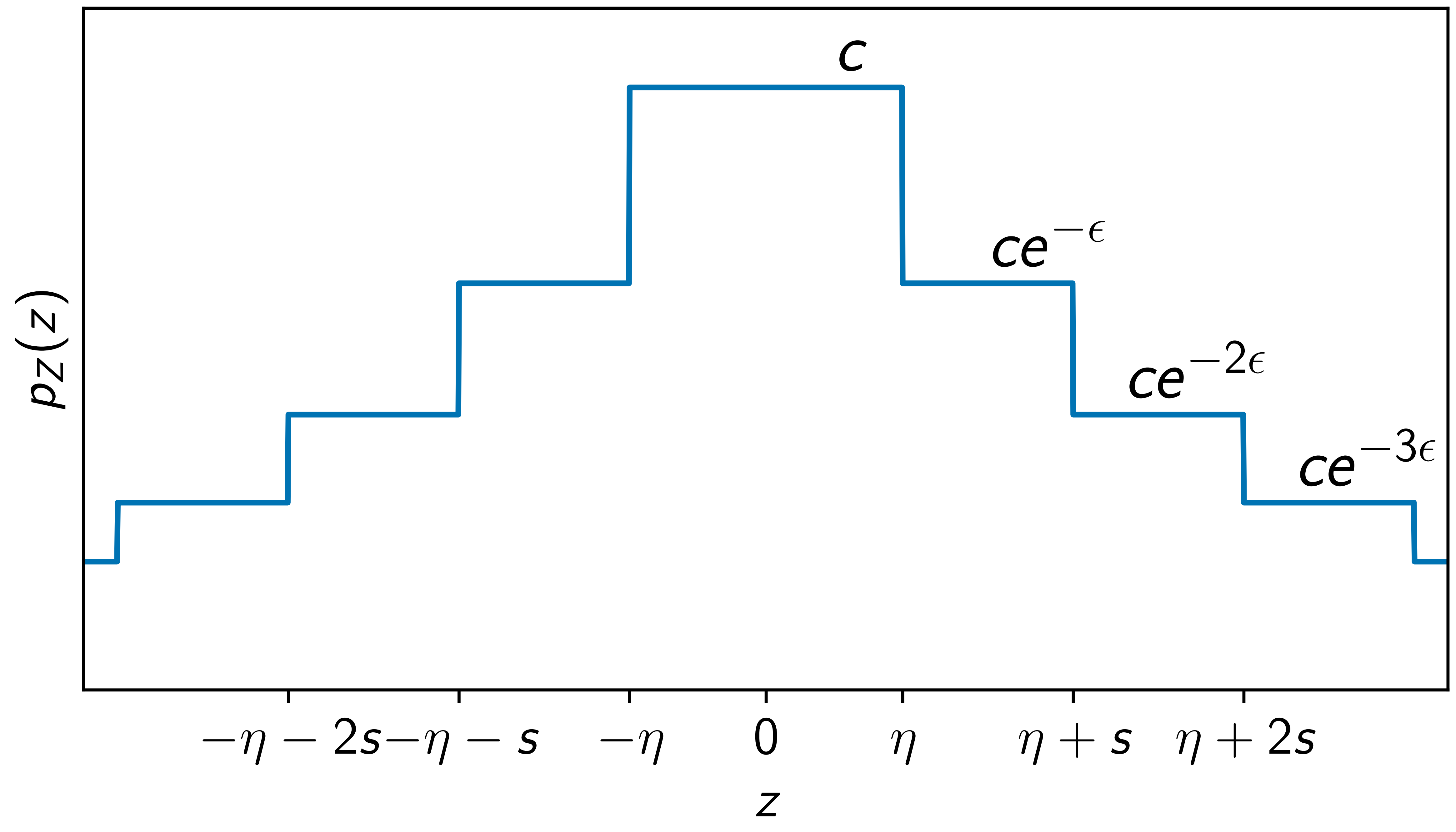
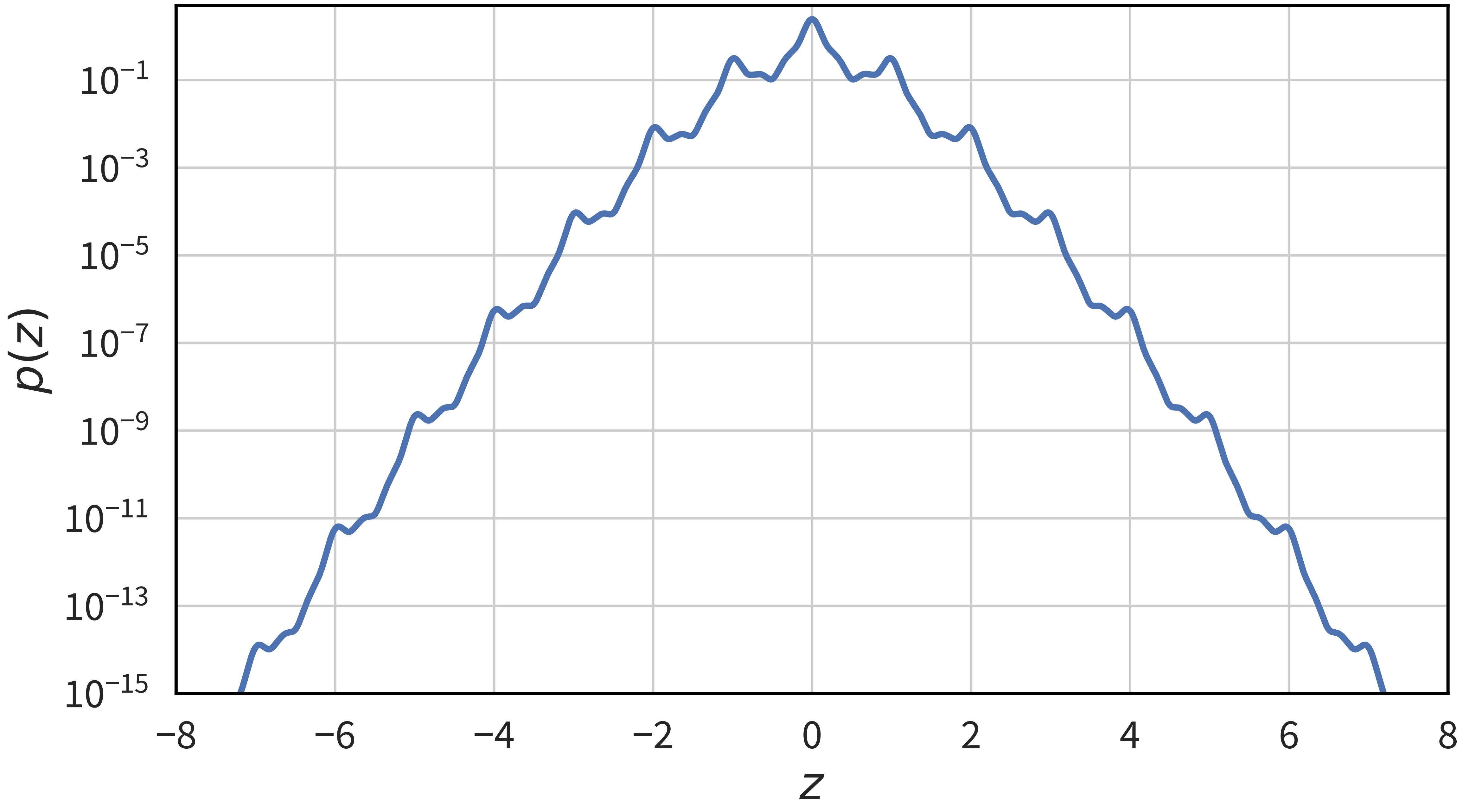
“Optimal” noise distributions

Beyond Gaussian and Laplace





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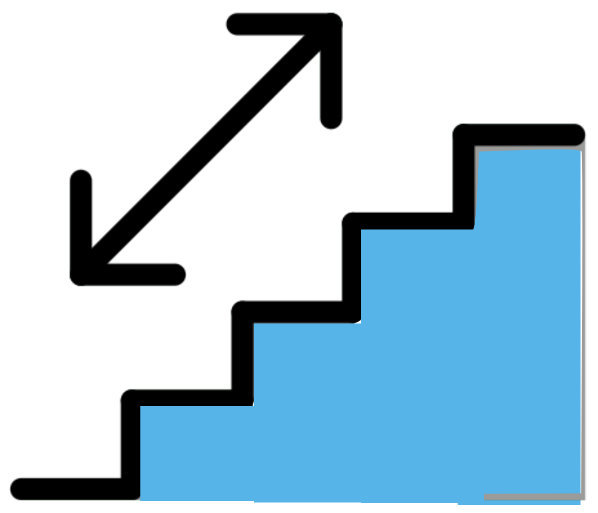

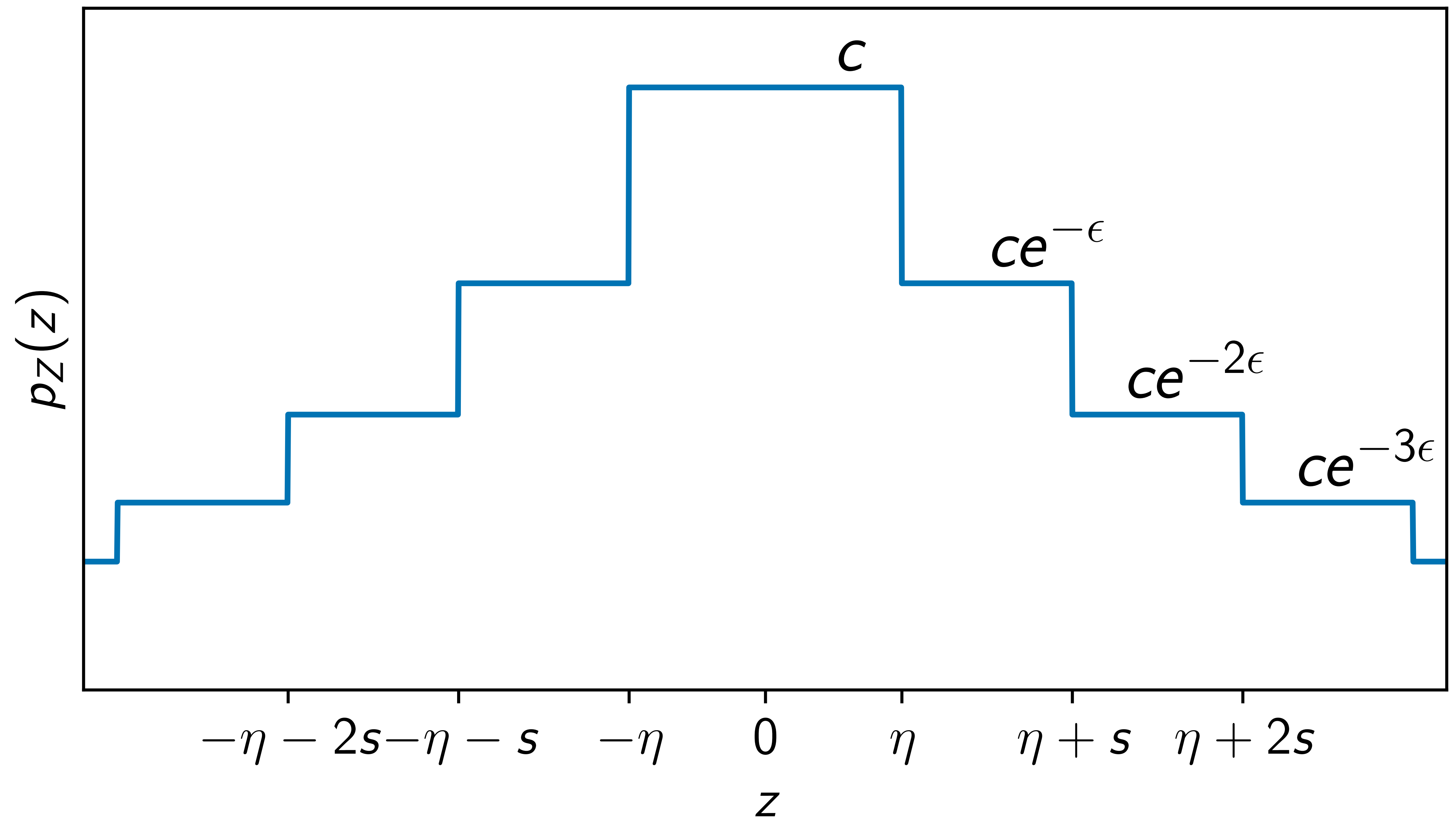

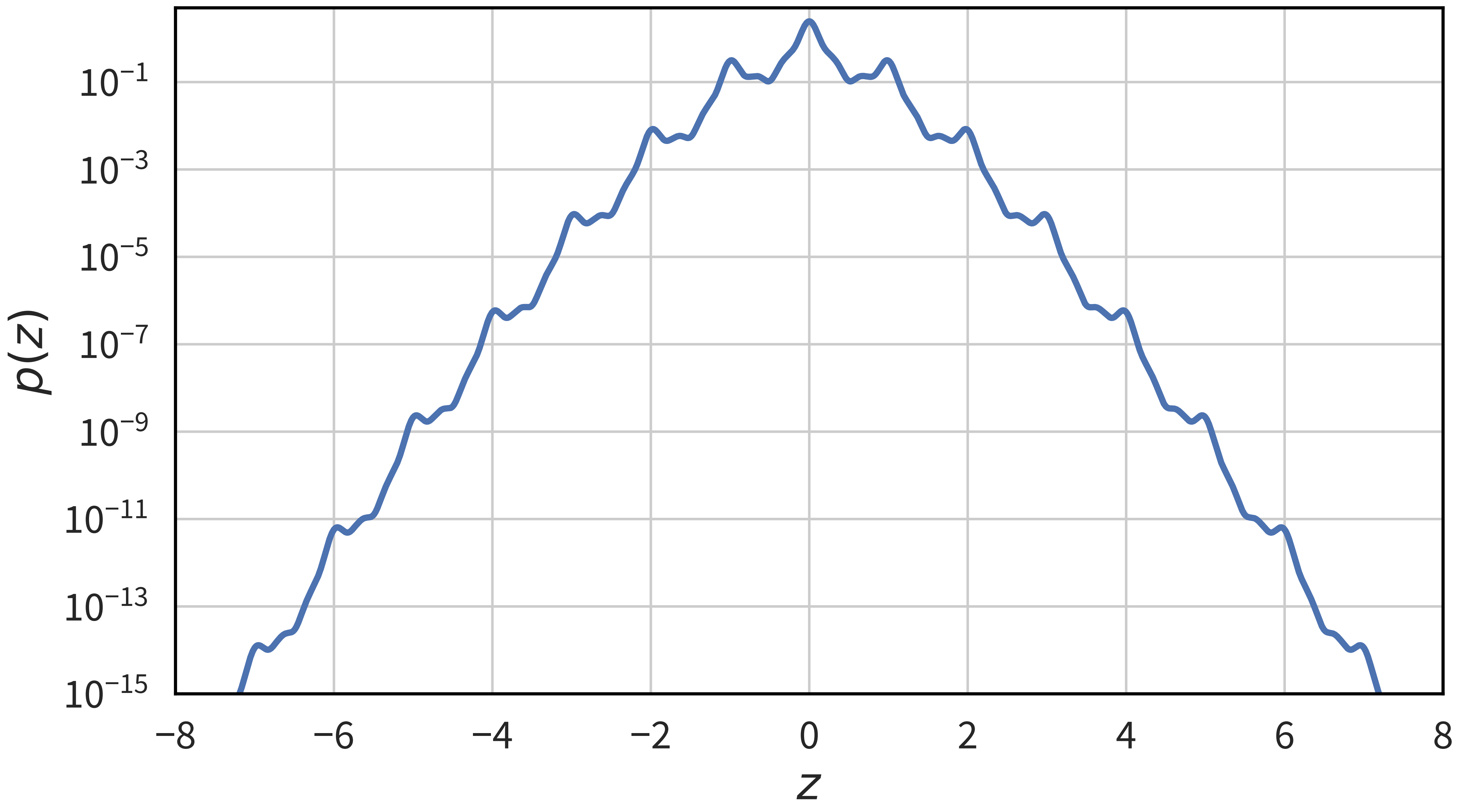

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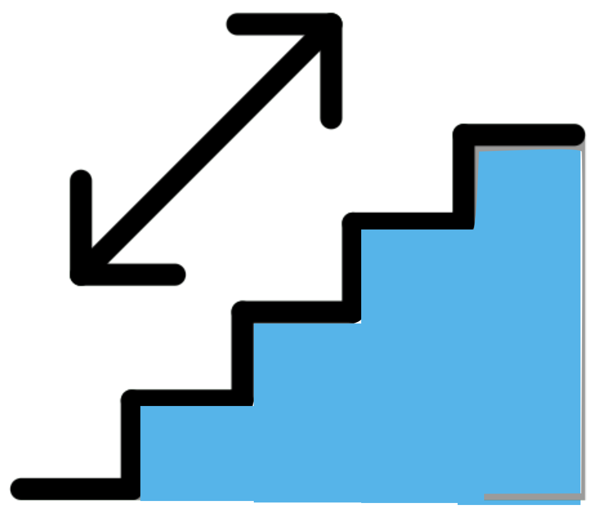
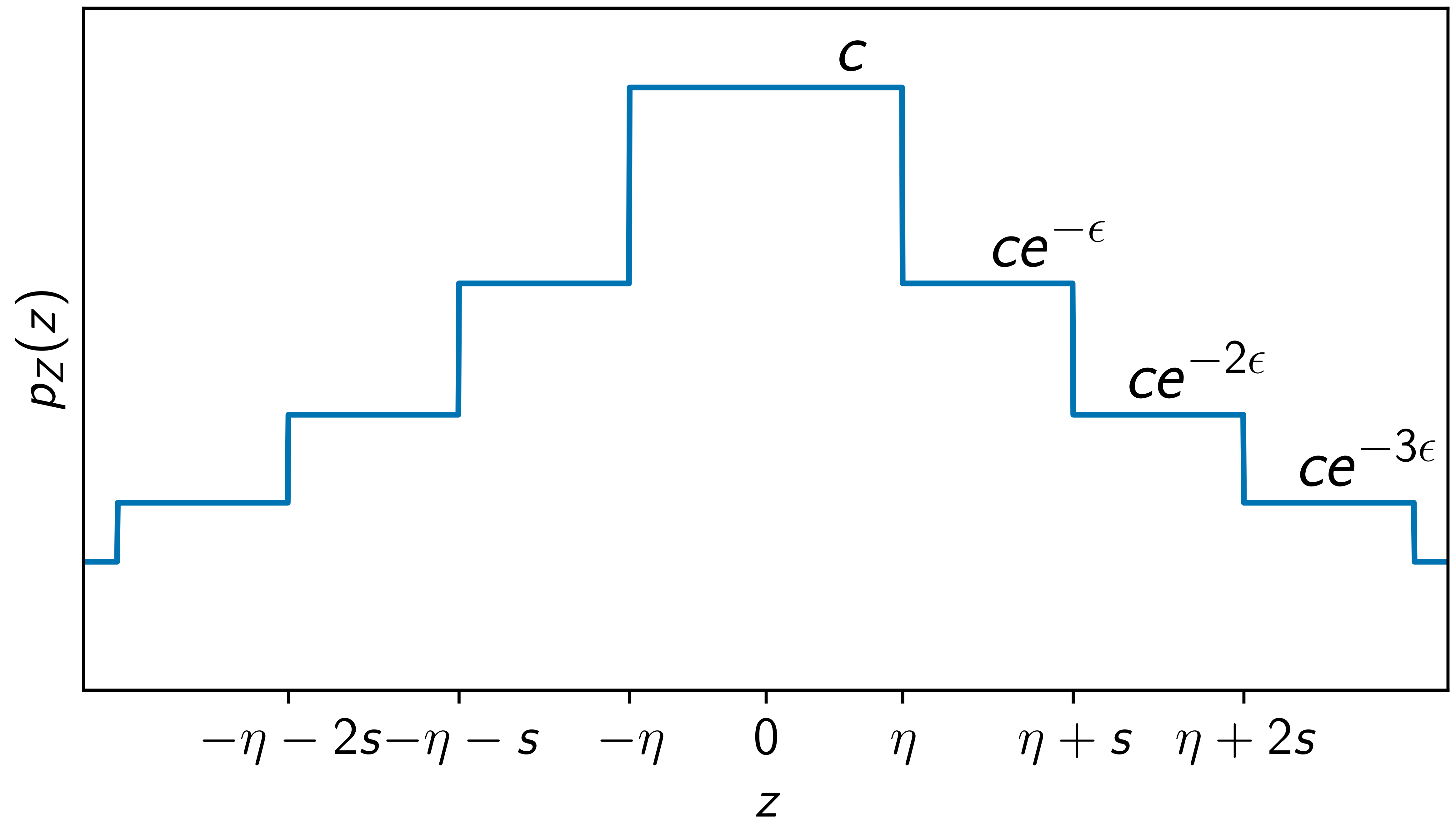
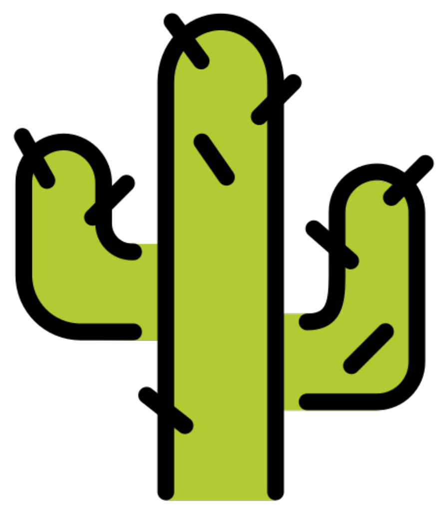
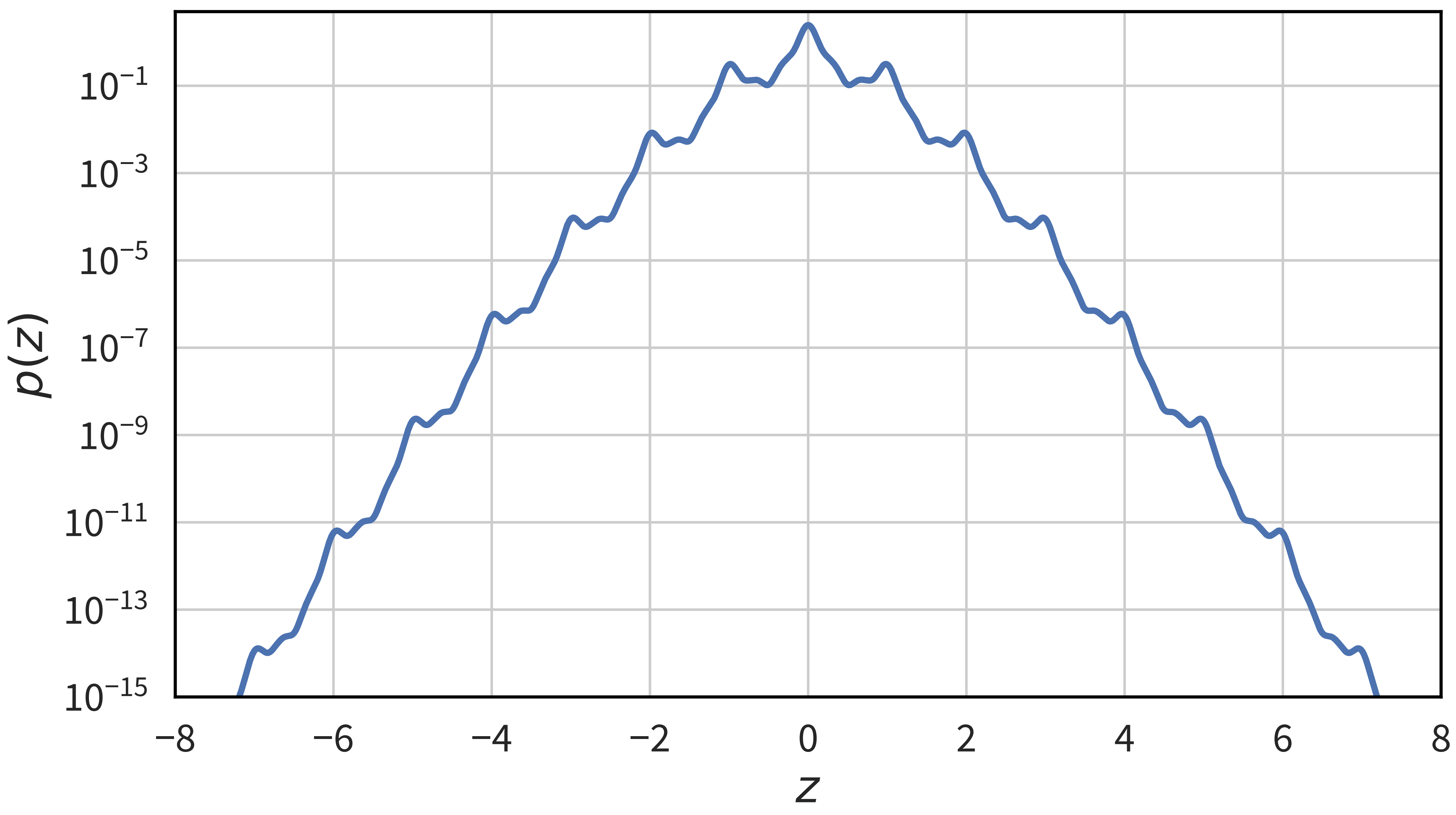


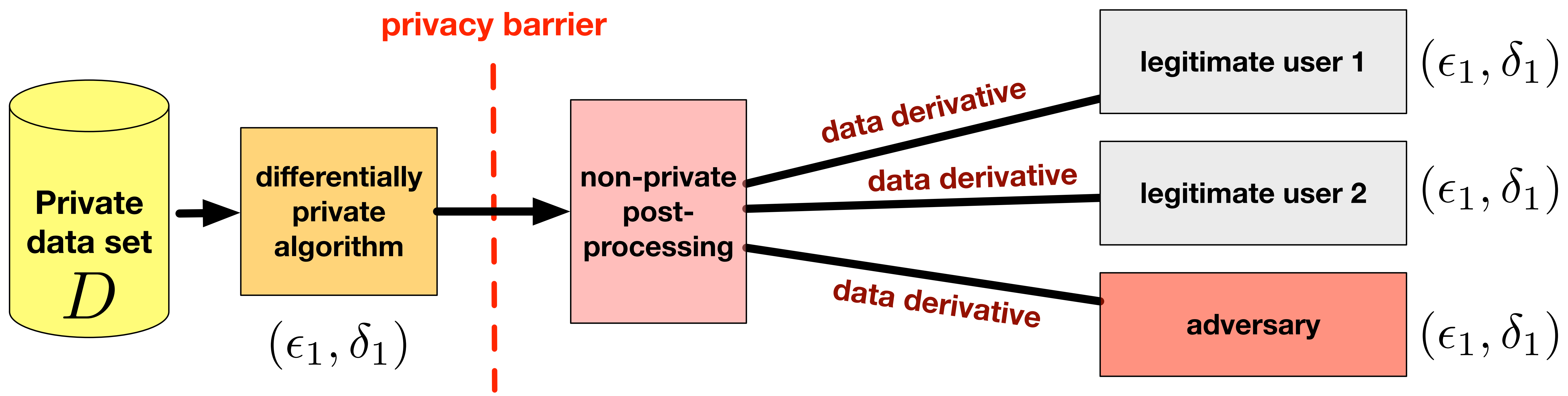






Post-processing invariance and composition

Nice properties of differential privacy



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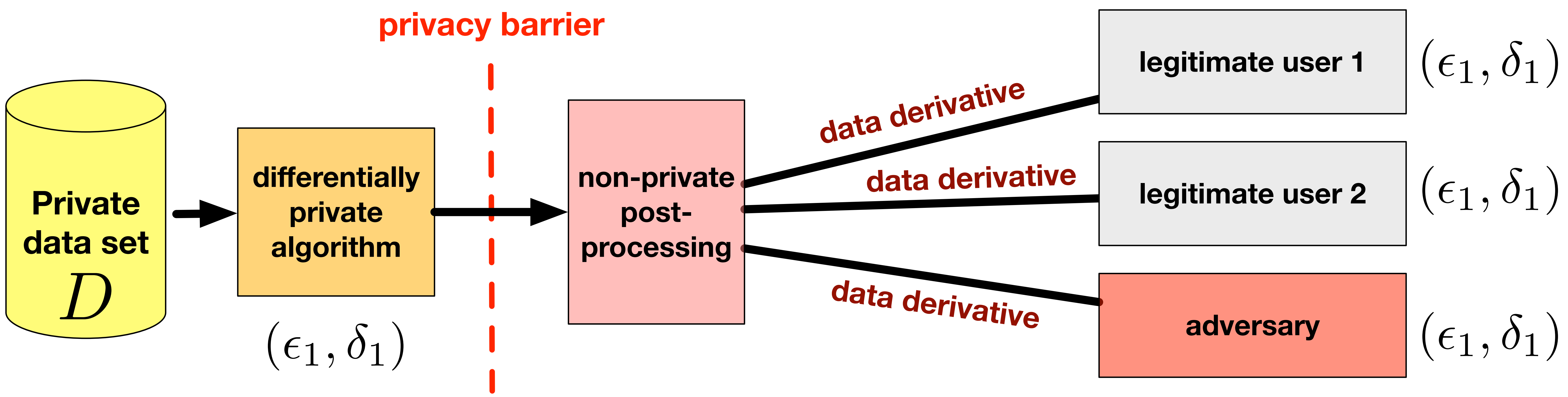

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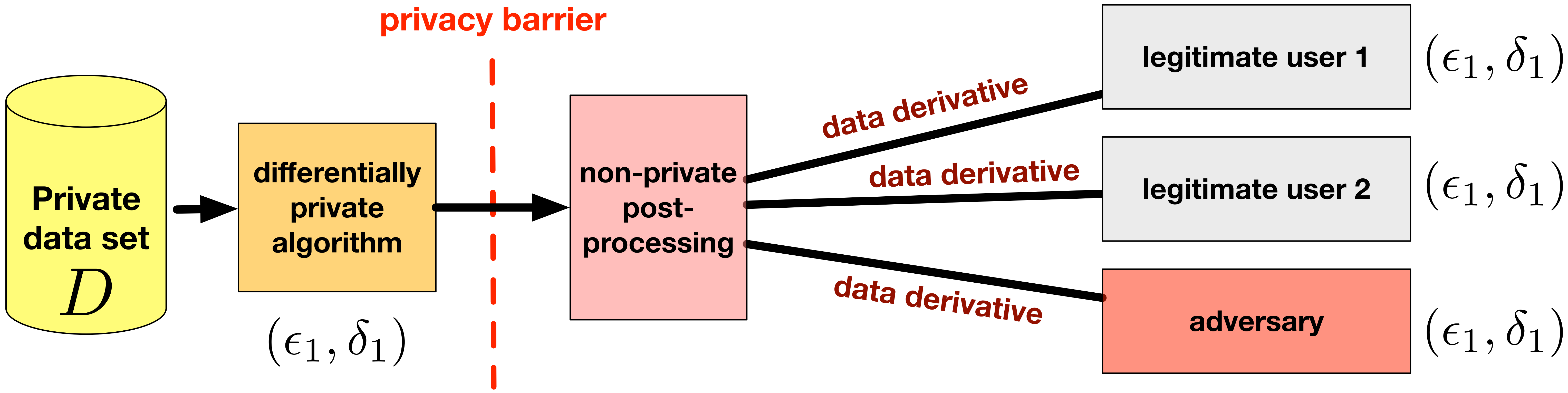

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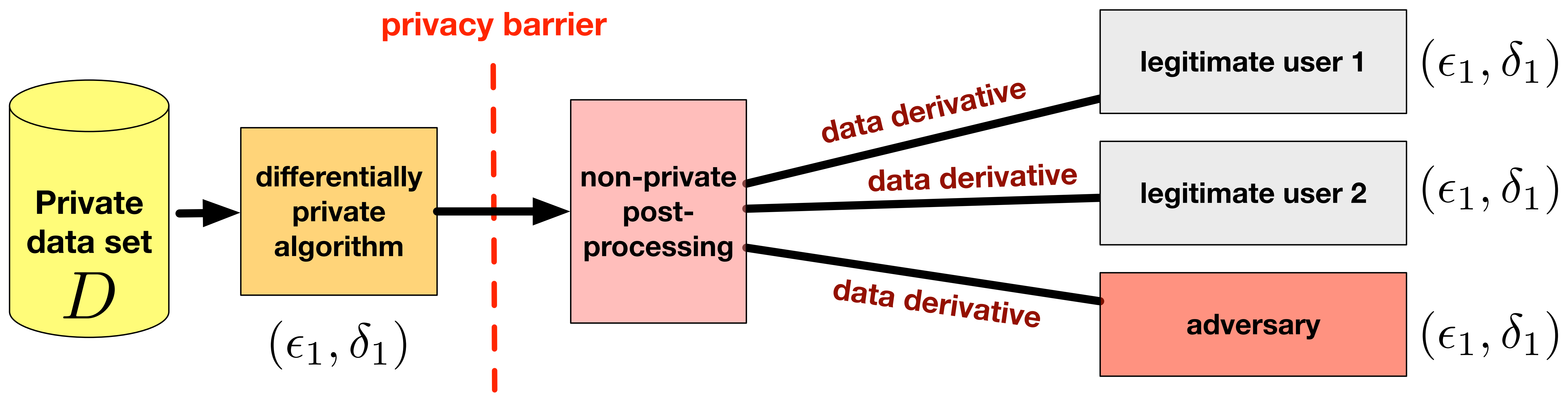

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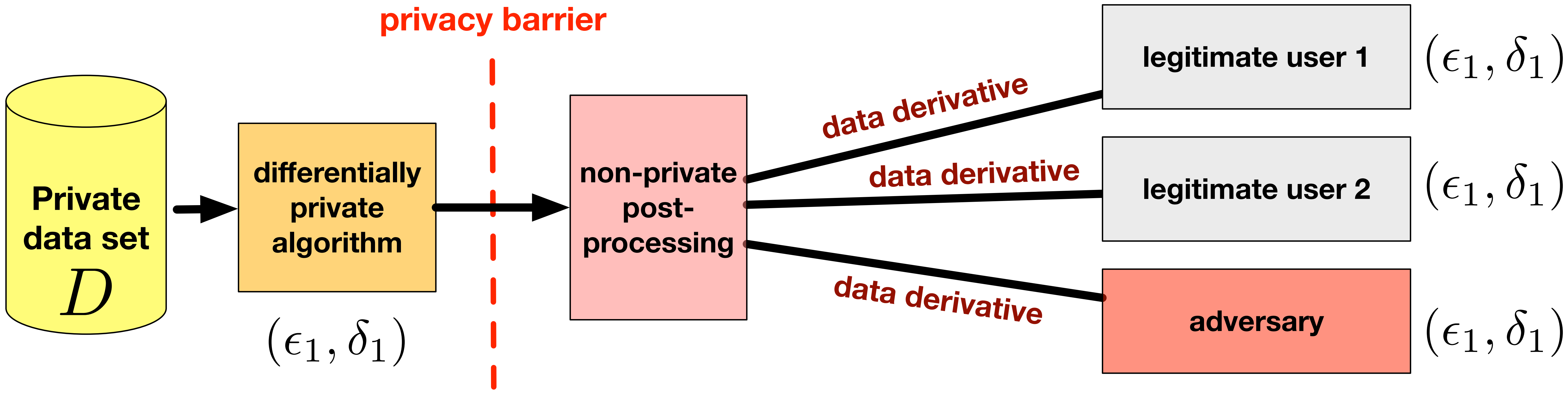

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Umezawa in Sagami Province

相州梅沢庄

Soshū Umezawanoshō

Vista 3

f-divergences/composition

Privacy loss random variables

Characterizing the distribution of the LLR

Sommer, Meisner, Mohammadi (2020), Zhu, Dong, Wang (2022)

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A challenge: this is defined for a single pair of inputs (x, x'). We would like to only deal with the "worst case" pair of inputs.

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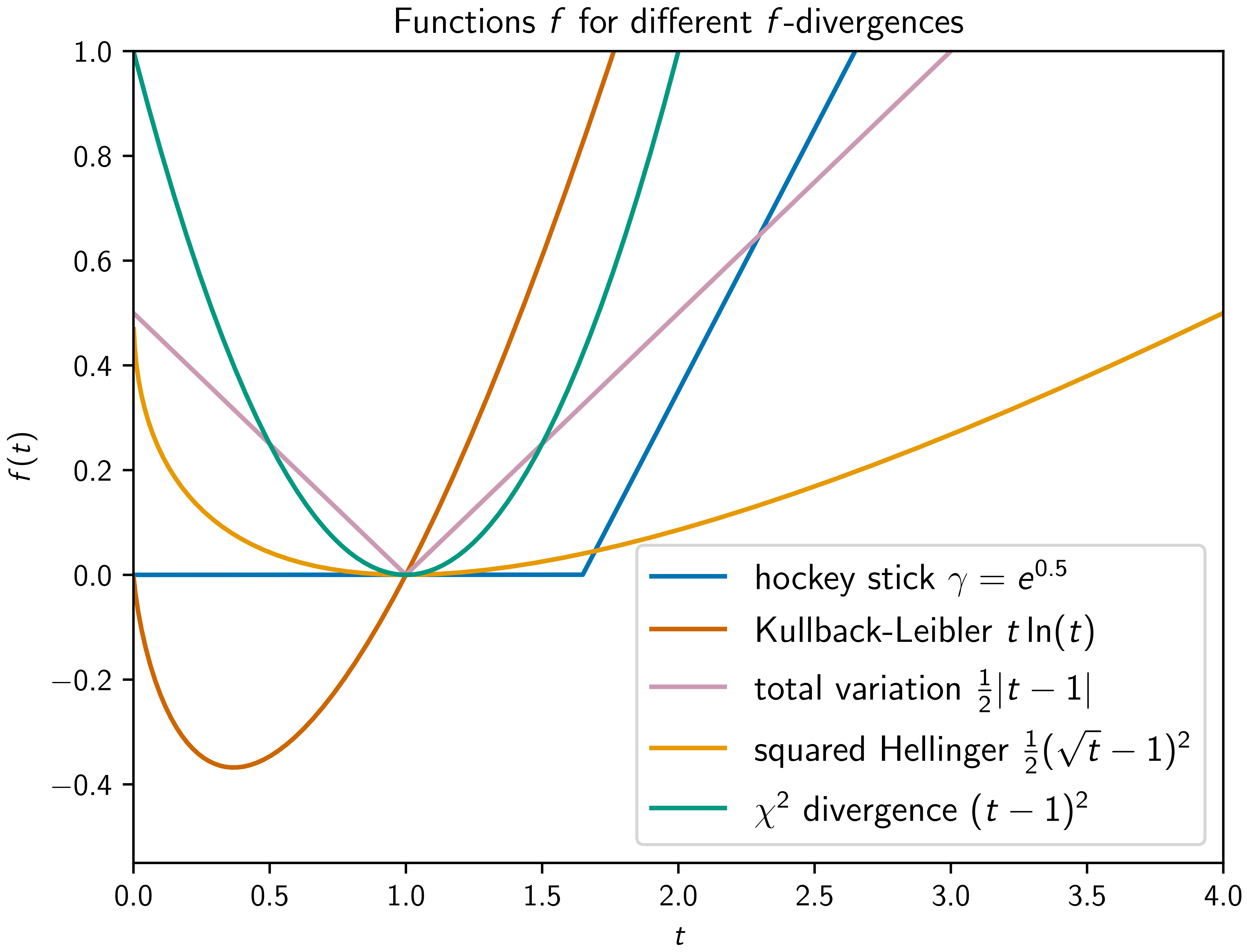
Generalized divergences and the 🏒 divergence

How different are these two distributions?



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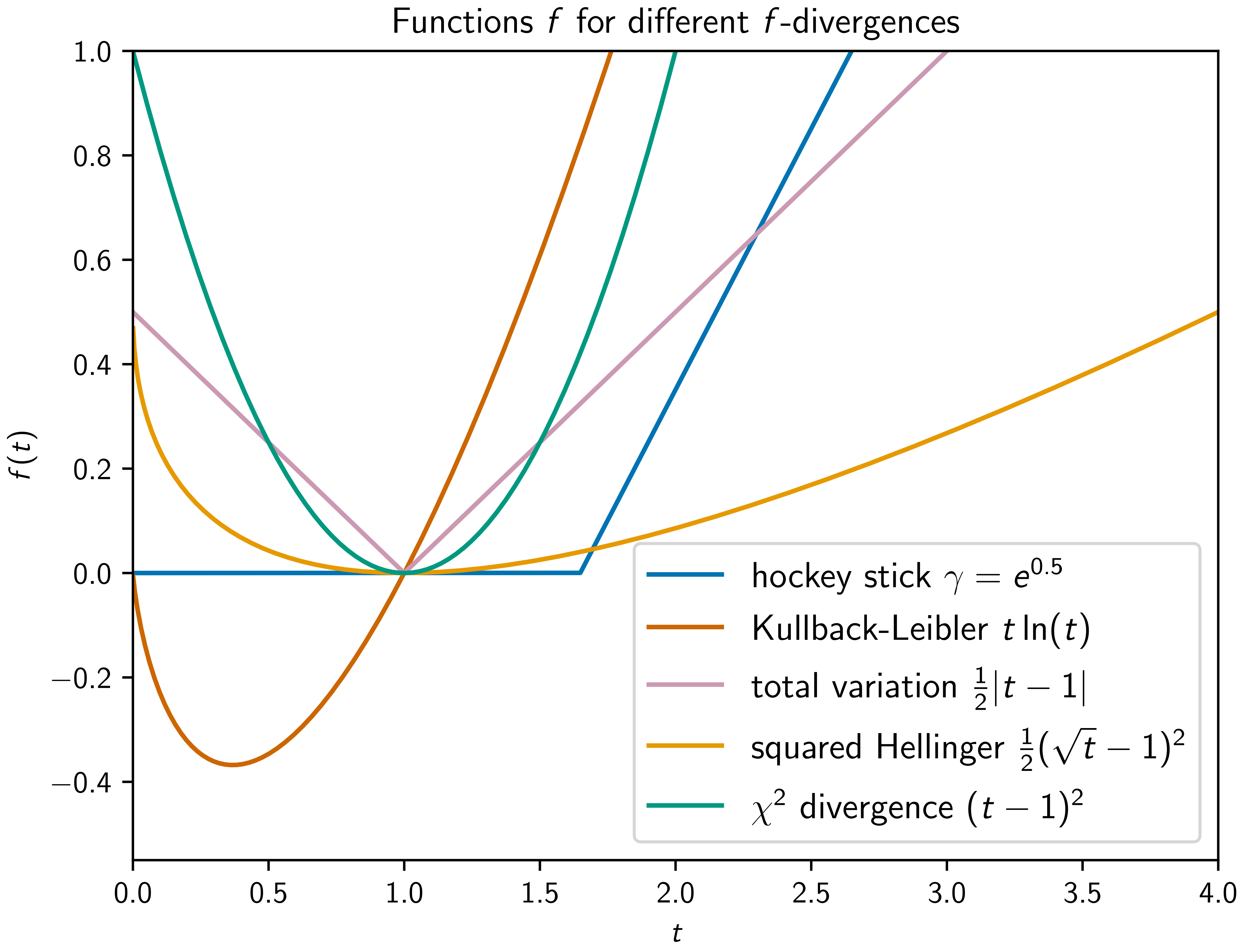

More generally, look at f-divergences: for any convex $f(\,\cdot\,)$,

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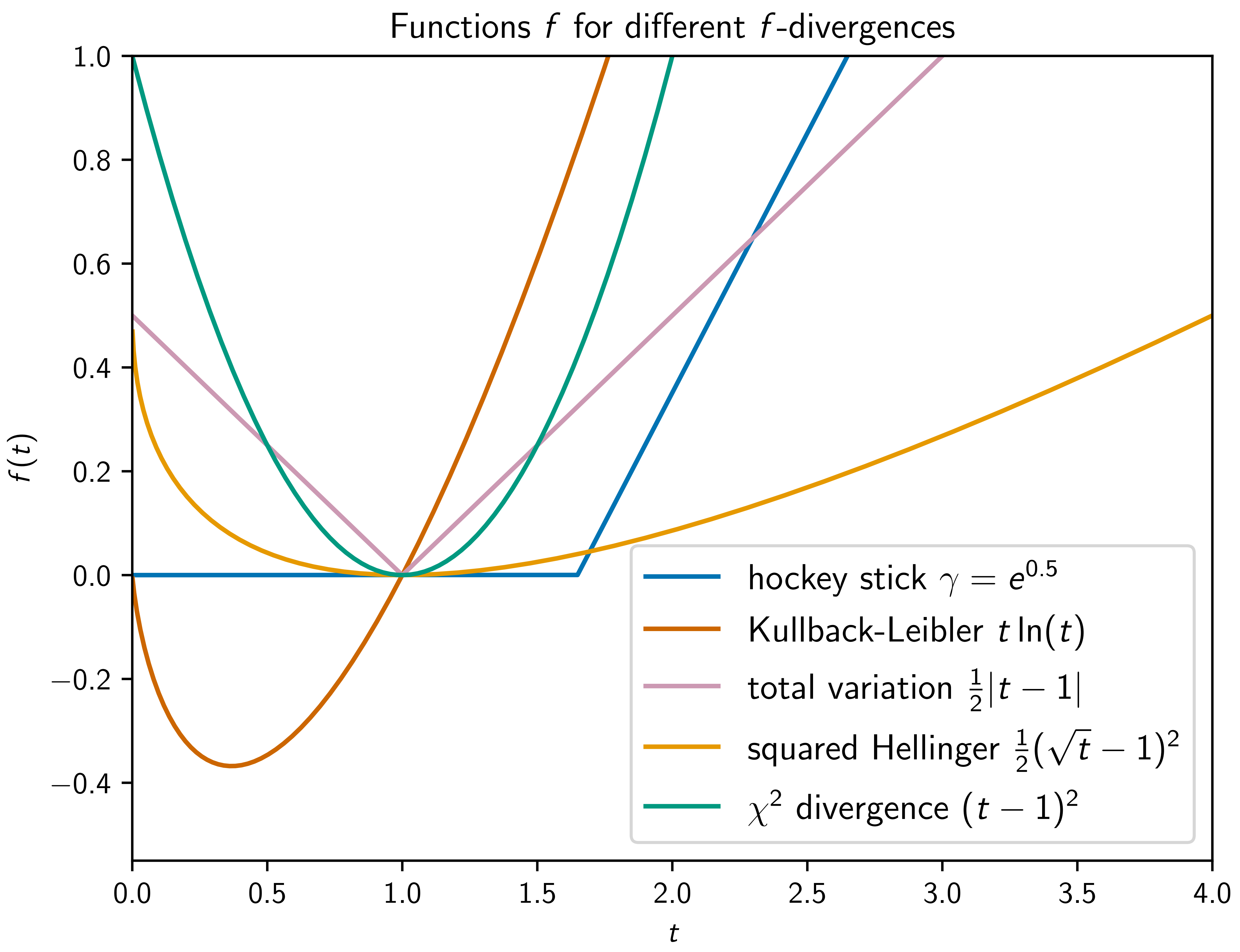

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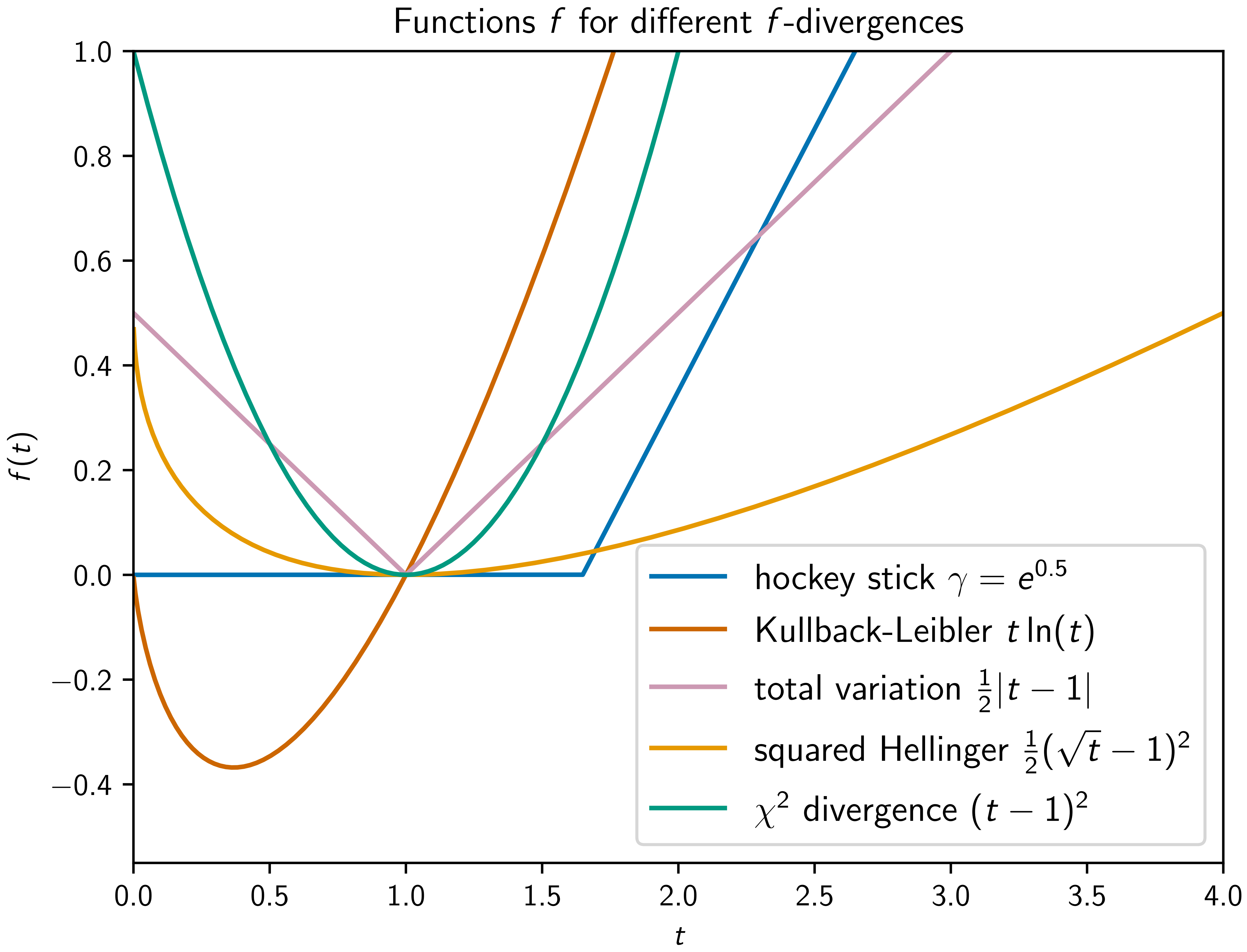

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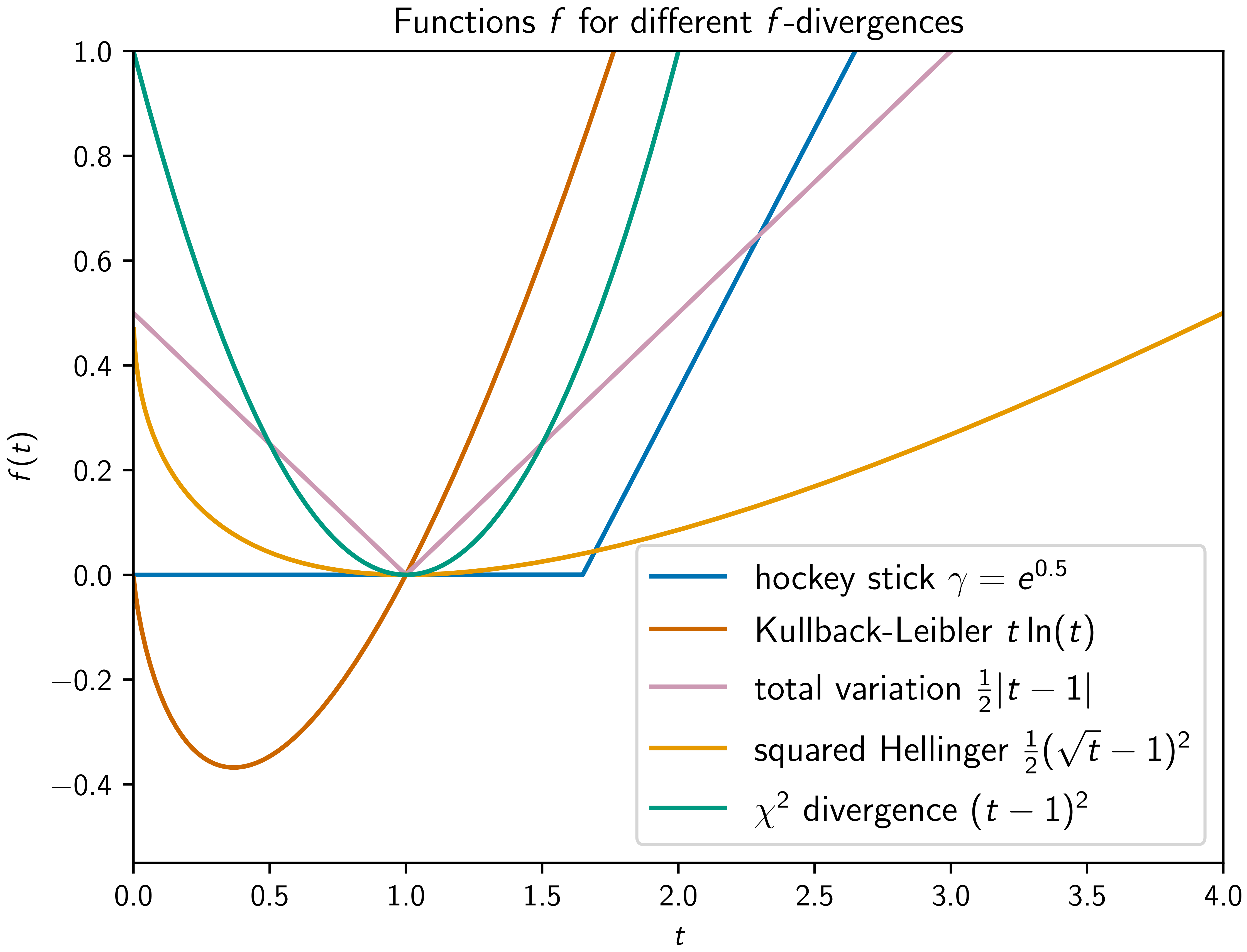

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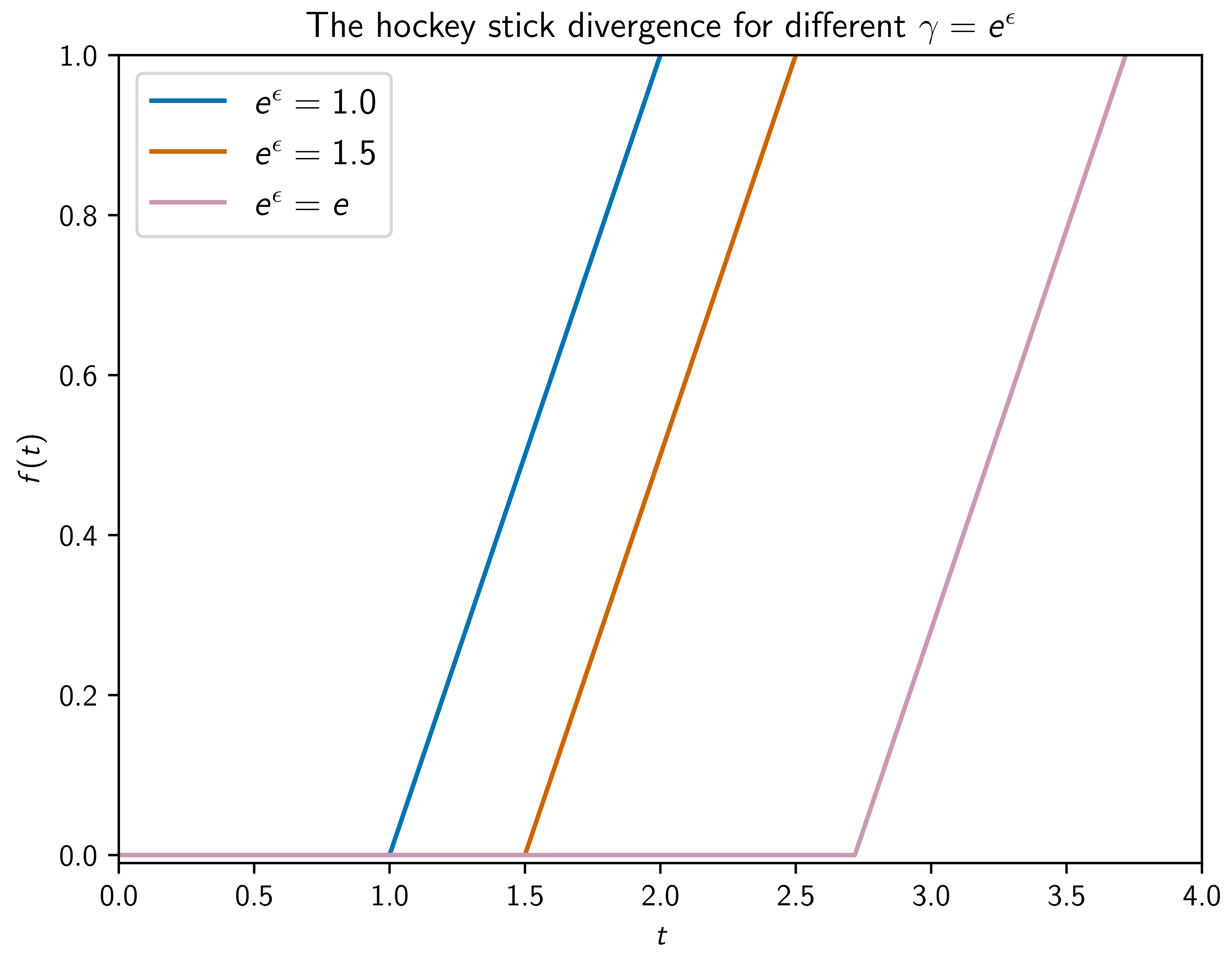
The PLRV and divergences

Interpreting DP as an f-divergence



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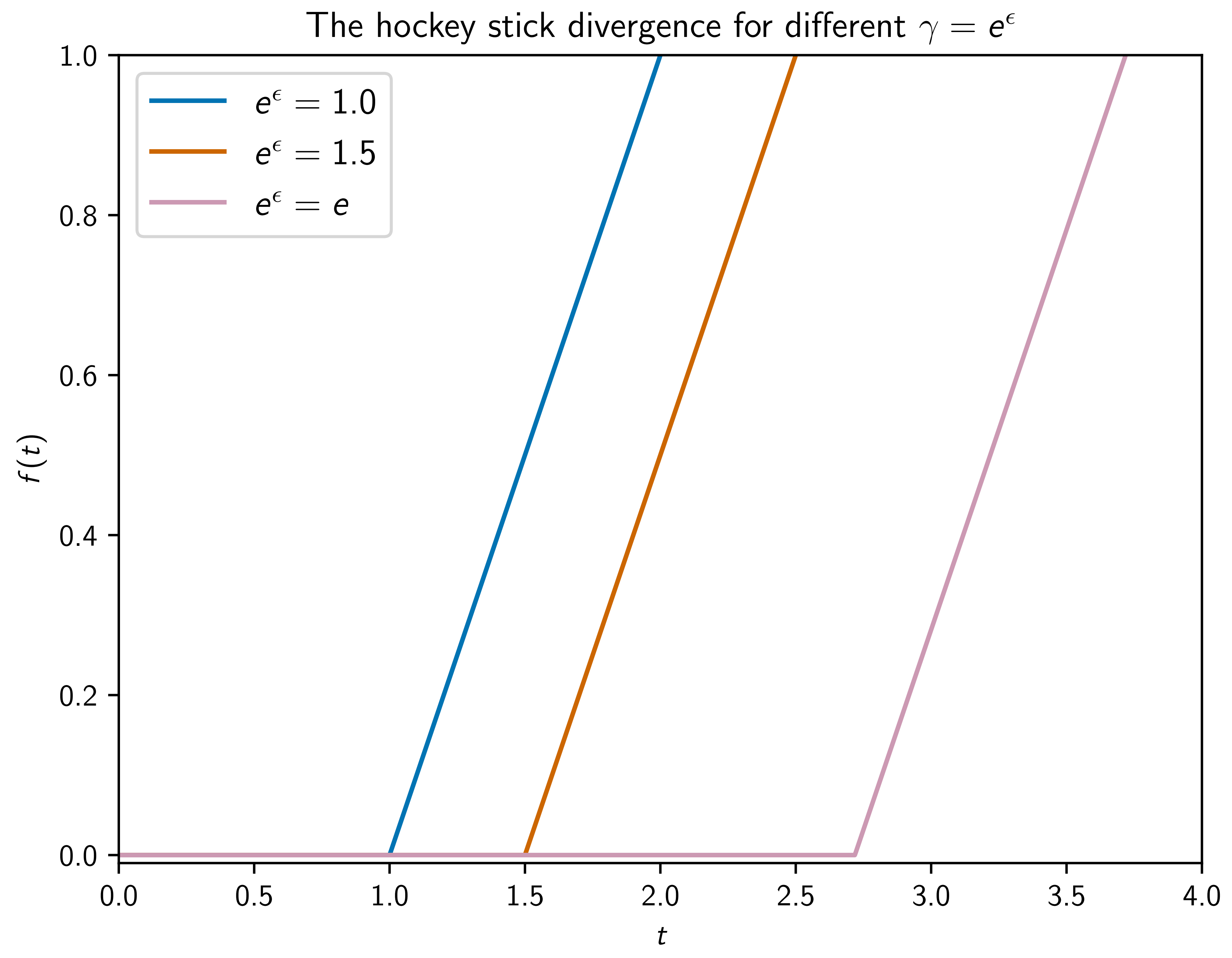

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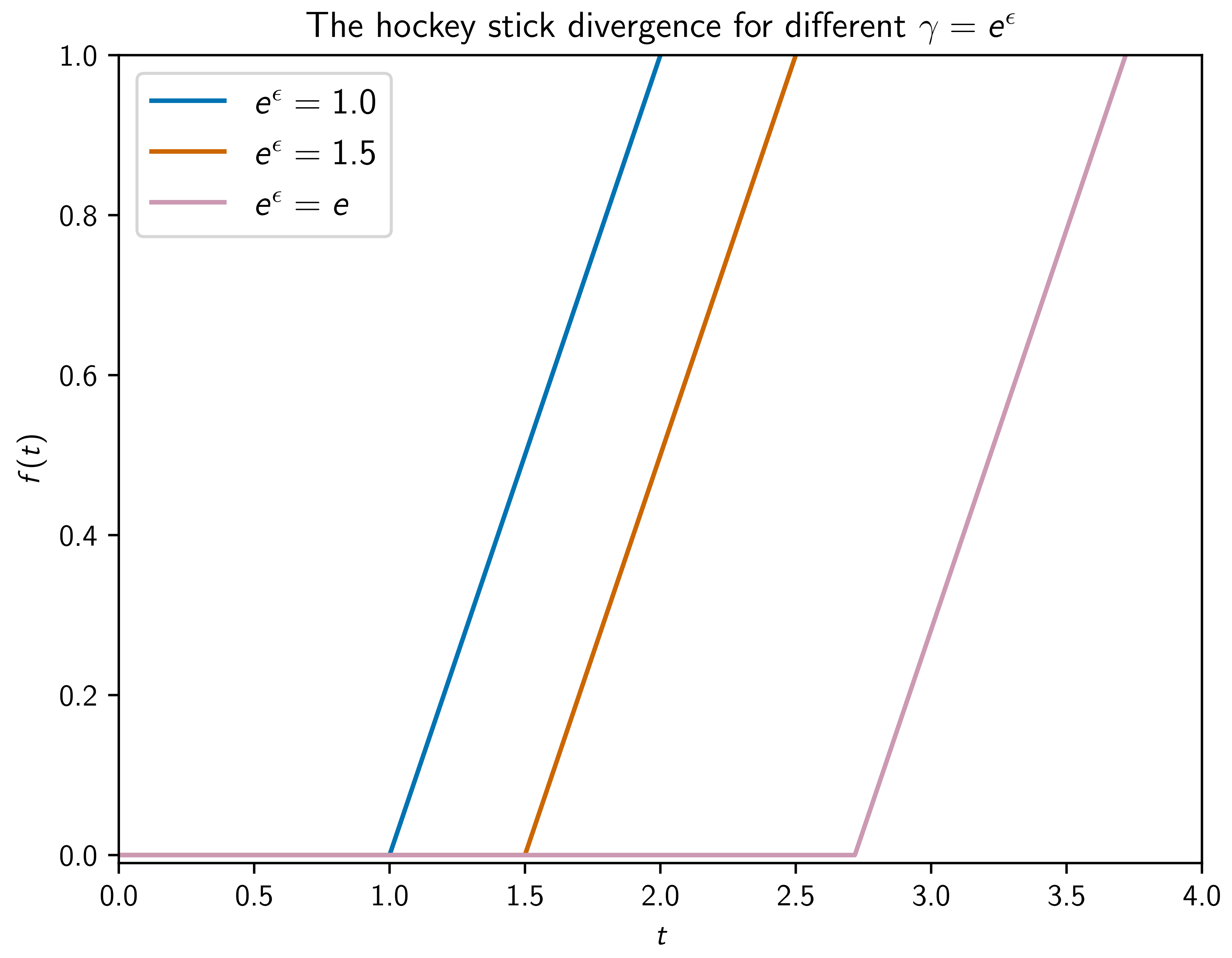

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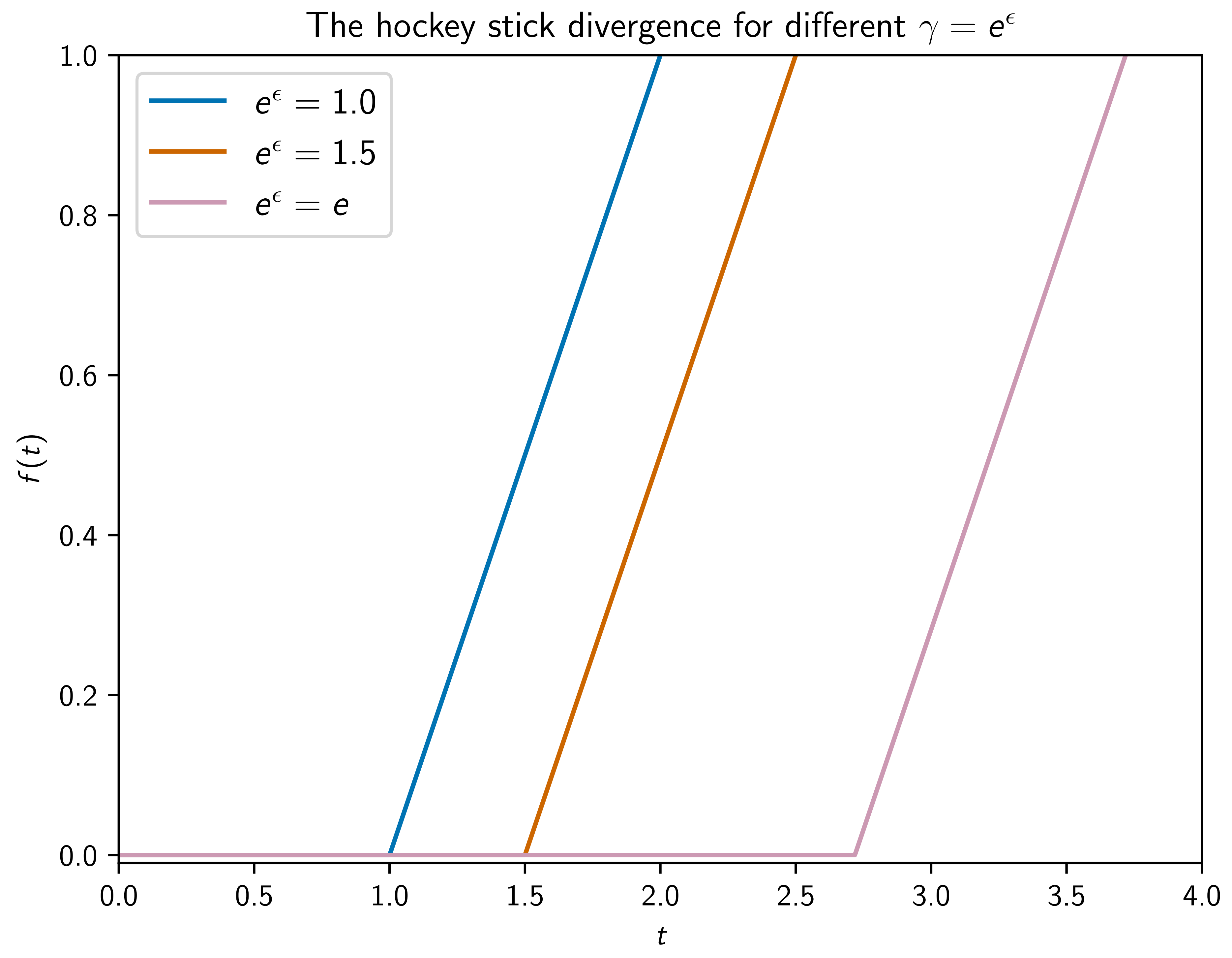

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Where L is the PLRV corresponding to (μ, ν).

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- We can use these dominating pairs to bound the loss for compositions.

Zhu, Dong, Wang (2022)

Composition and divergences

Adding things up

With thanks to Flavio Calmon, Oliver Kosut, and Shahab Asoodeh!

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- Numerical approaches

With thanks to Flavio Calmon, Oliver Kosut, and Shahab Asoodeh!

A taxonomy of composition analyses

(on non-interactive settings, also non-exhaustive)

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Lebeda, Regehr, Kamath, Steinke (2024)

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Tilting a distribution

Maintaining exactness for composition

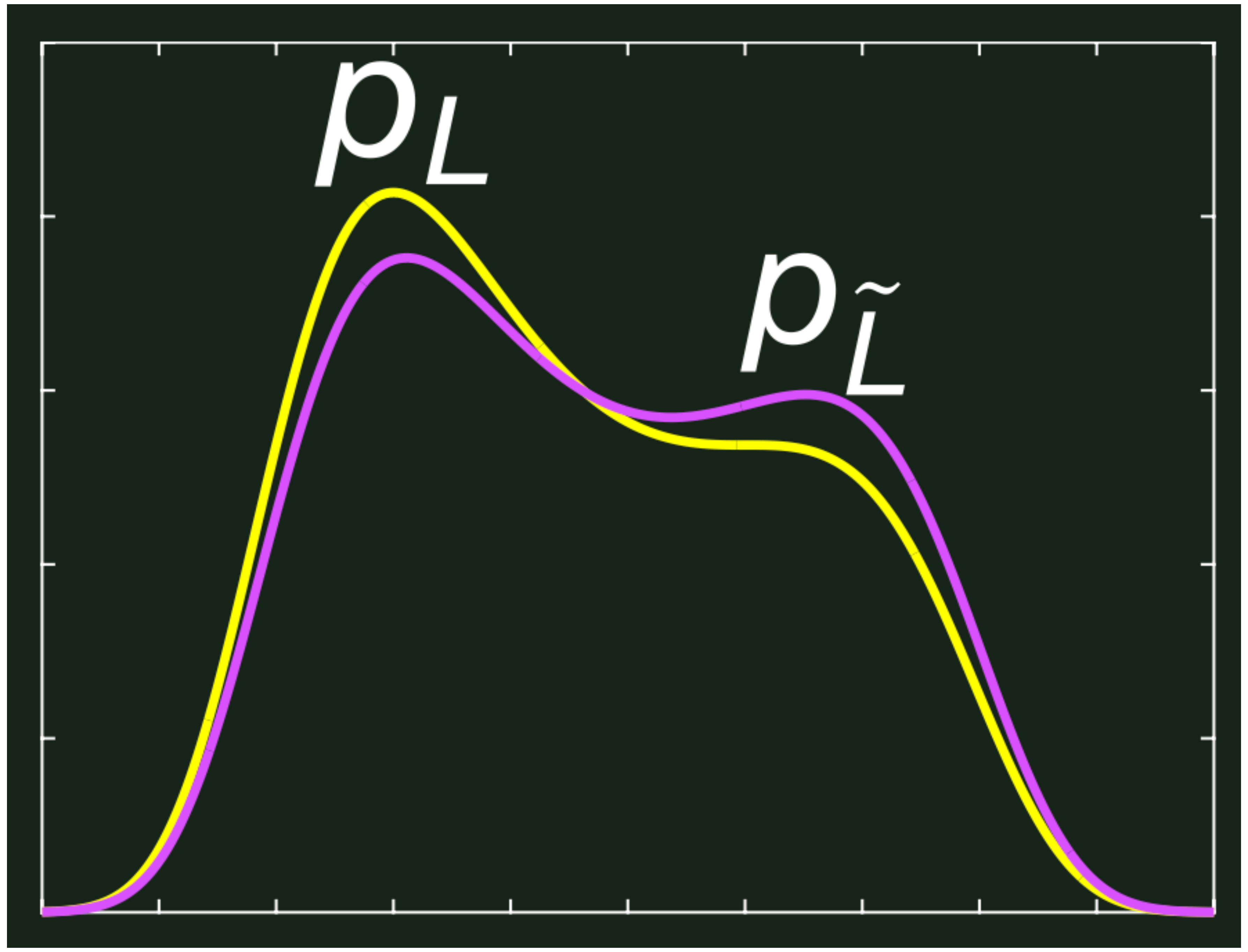

Figure: Oliver Kosut

Alghamdi, Gomez, Asoodeh, Calmon, Kosut, Sankar (2023)

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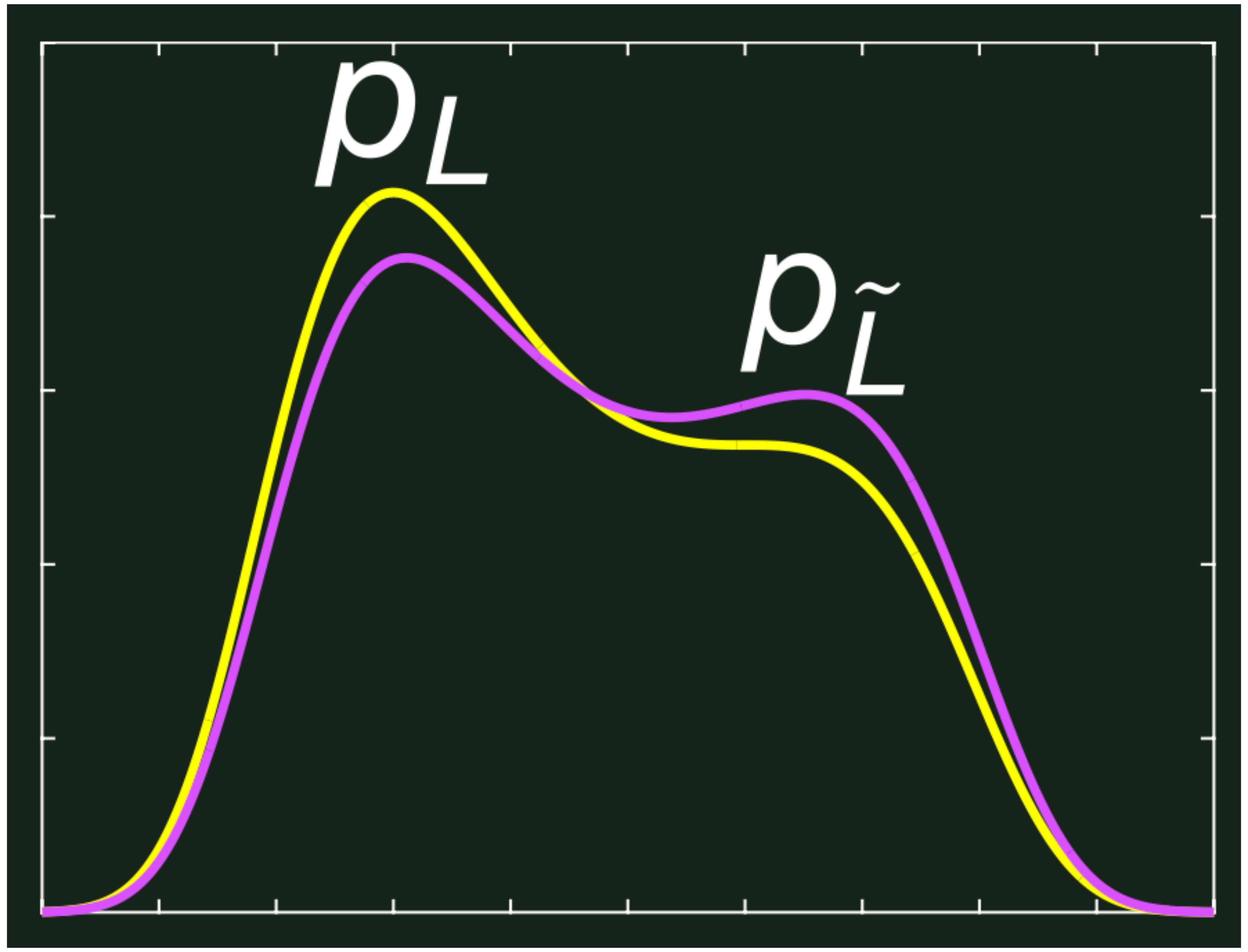

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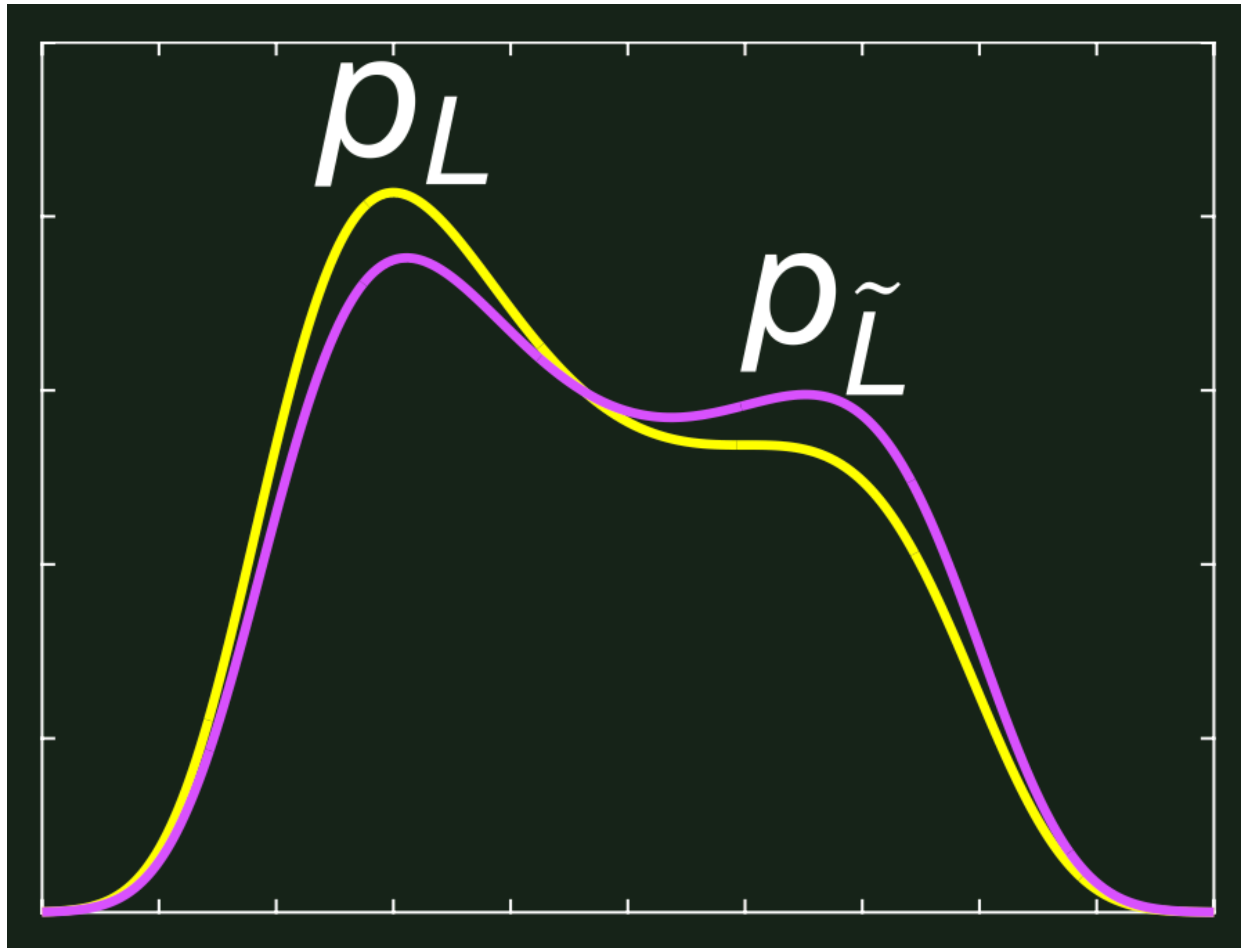

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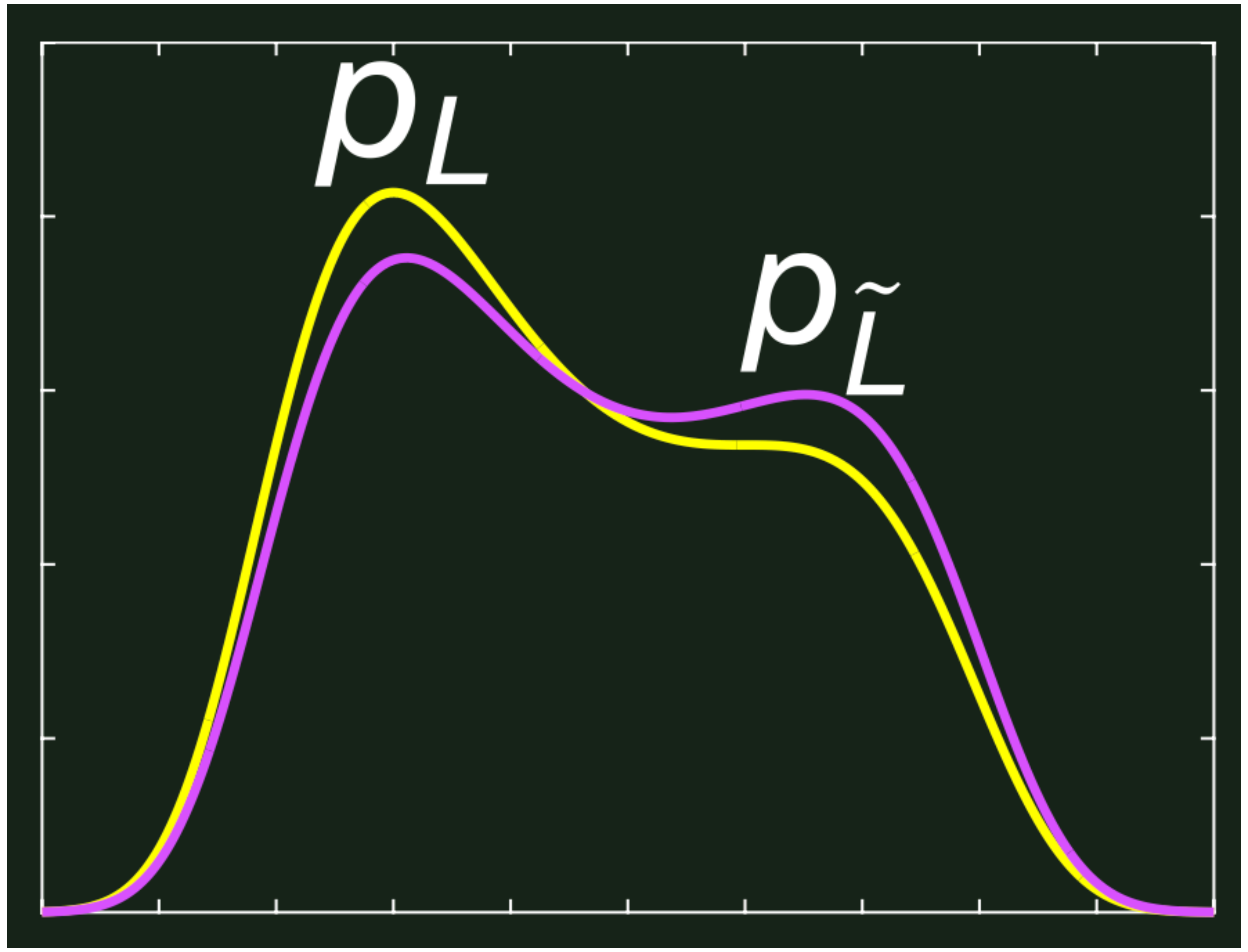

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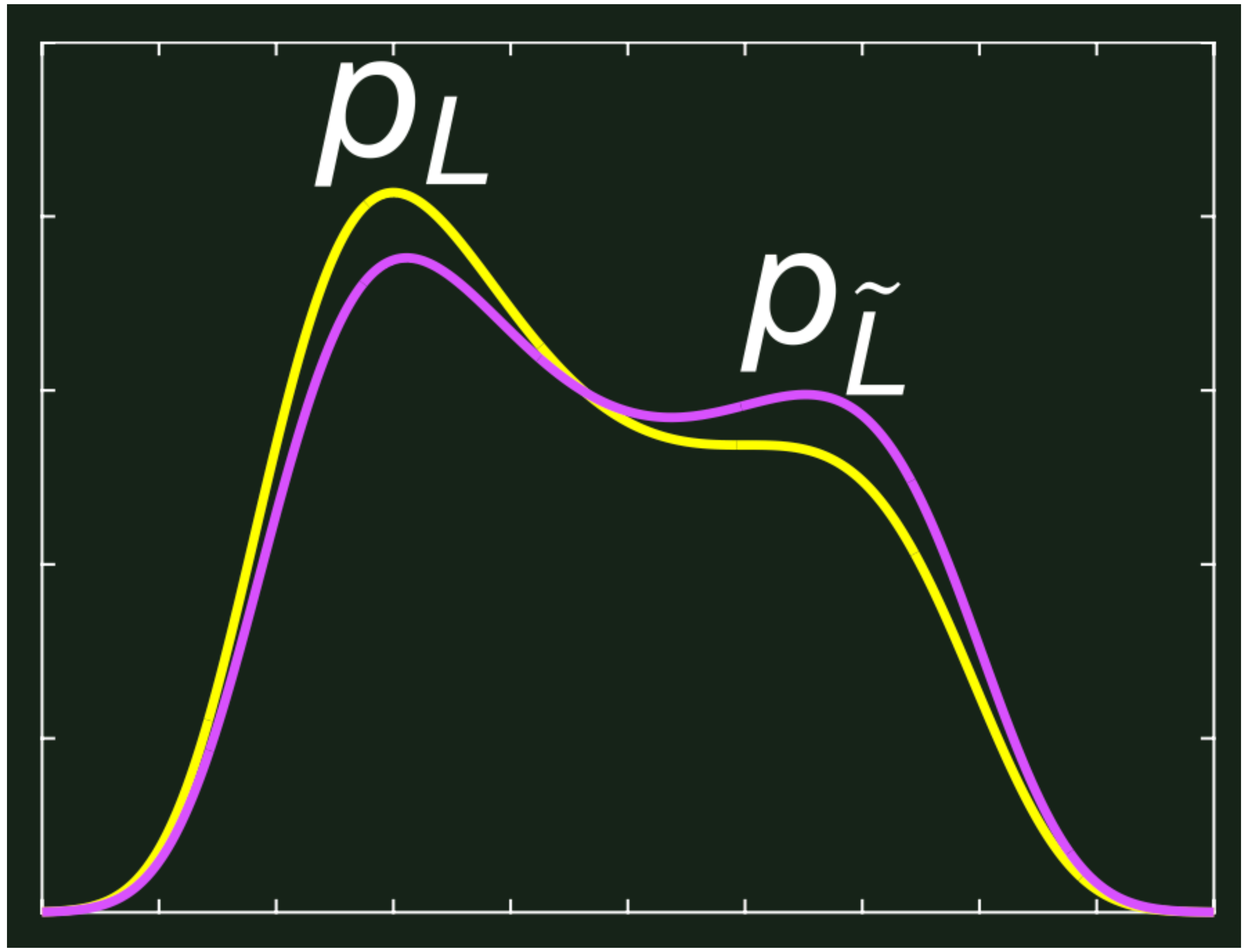

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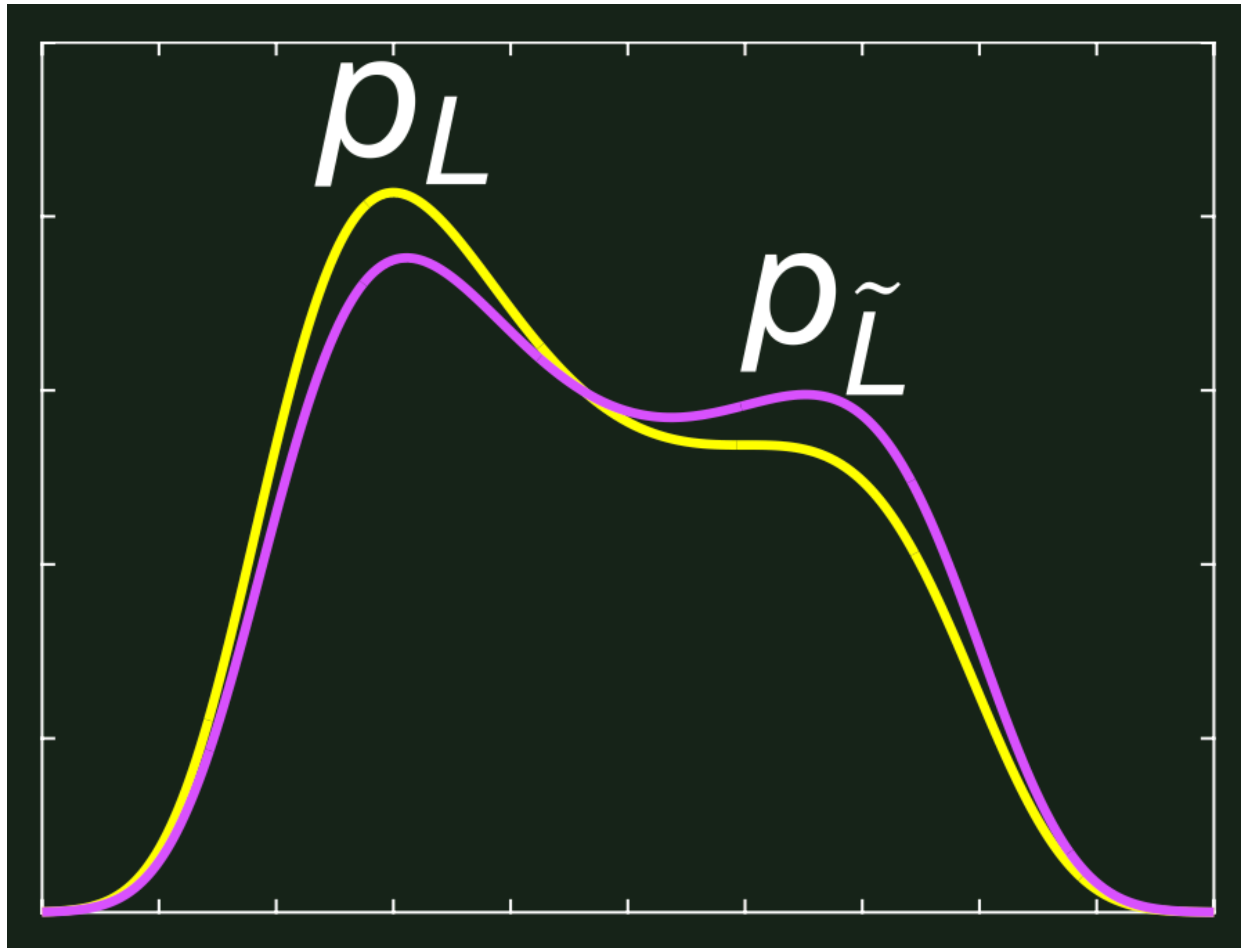

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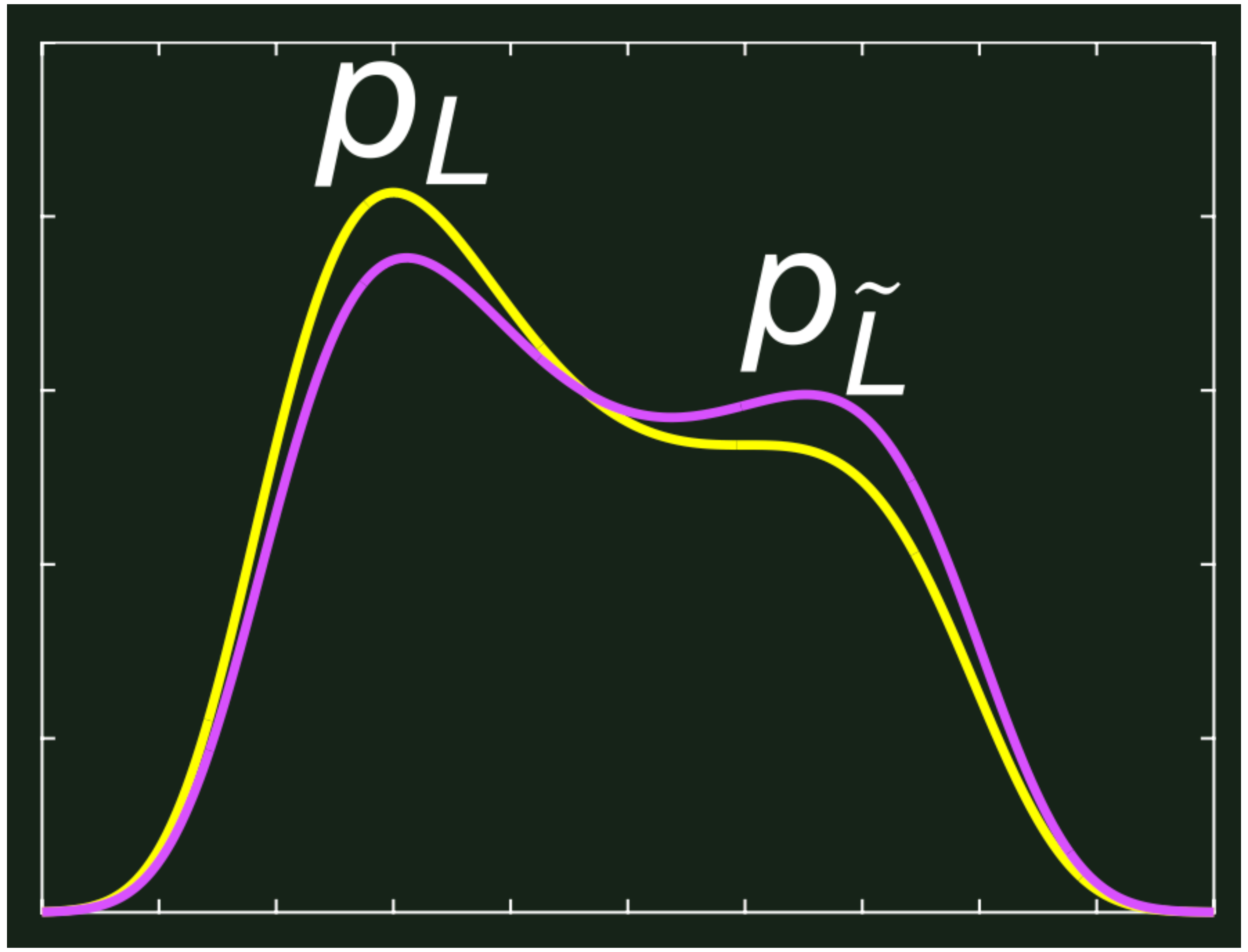

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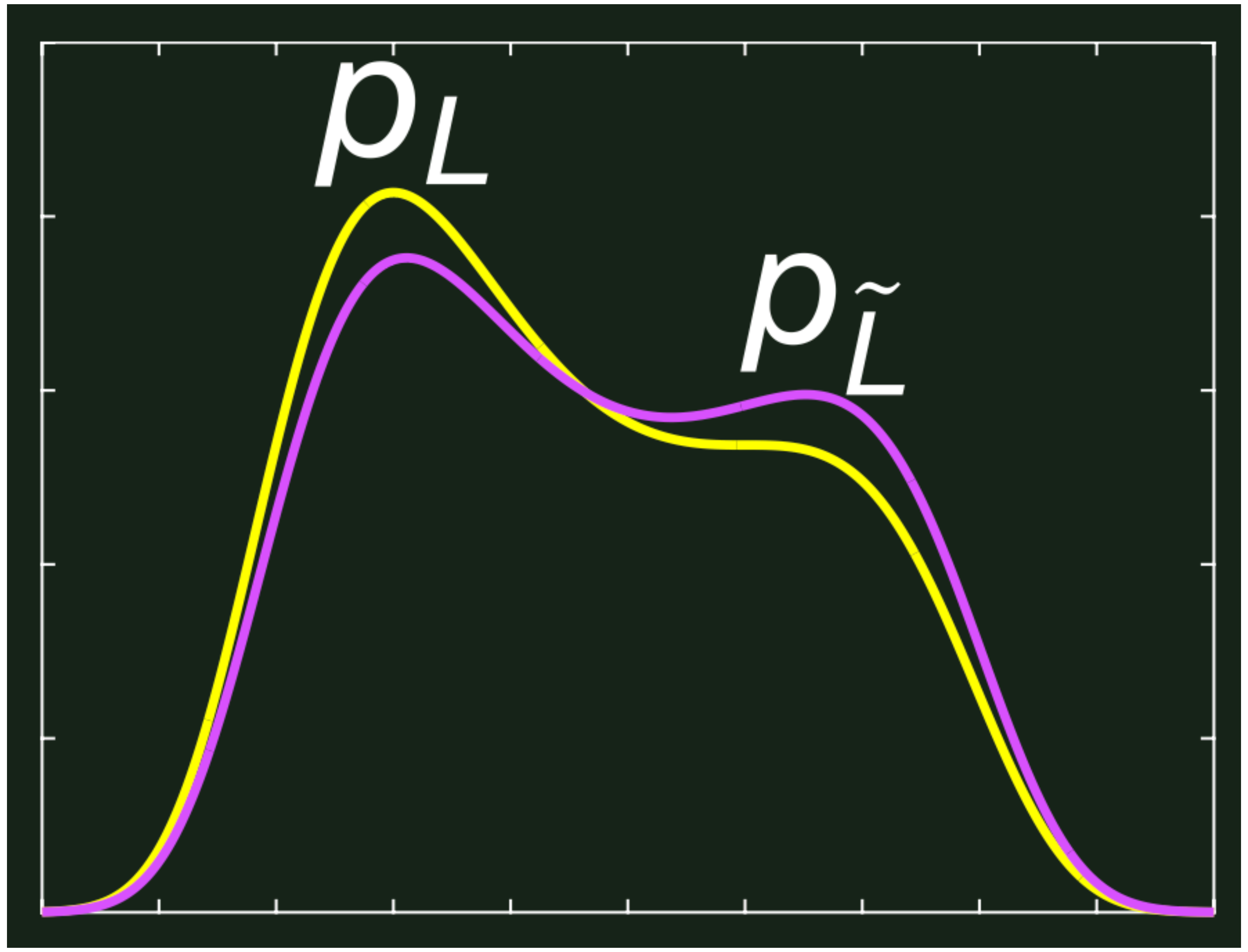

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Can use this to derive a **“saddle-point” accountant** in terms of the exponent.

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Tilting in other contexts

(Beyond *Don Quixote*)

A tiling (not tilting) by
M.C. Escher, not F. Esscher

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Perhaps of interest to folks here? Botev (2017) uses it to exact iid simulation from the truncated multivariate normal distribution.

Shichiri Beach in Sagami Province

相州七里浜

Soshū Shichiri-ga-hama

Vista 4

contraction coefficients/iteration

Maximum likelihood and ERM

Optimization and privacy

[Chaudhuri, Monteleoni, Sarwate 2011] [Zhang, Zhang, Xiao, Yang, Winslett 2012]

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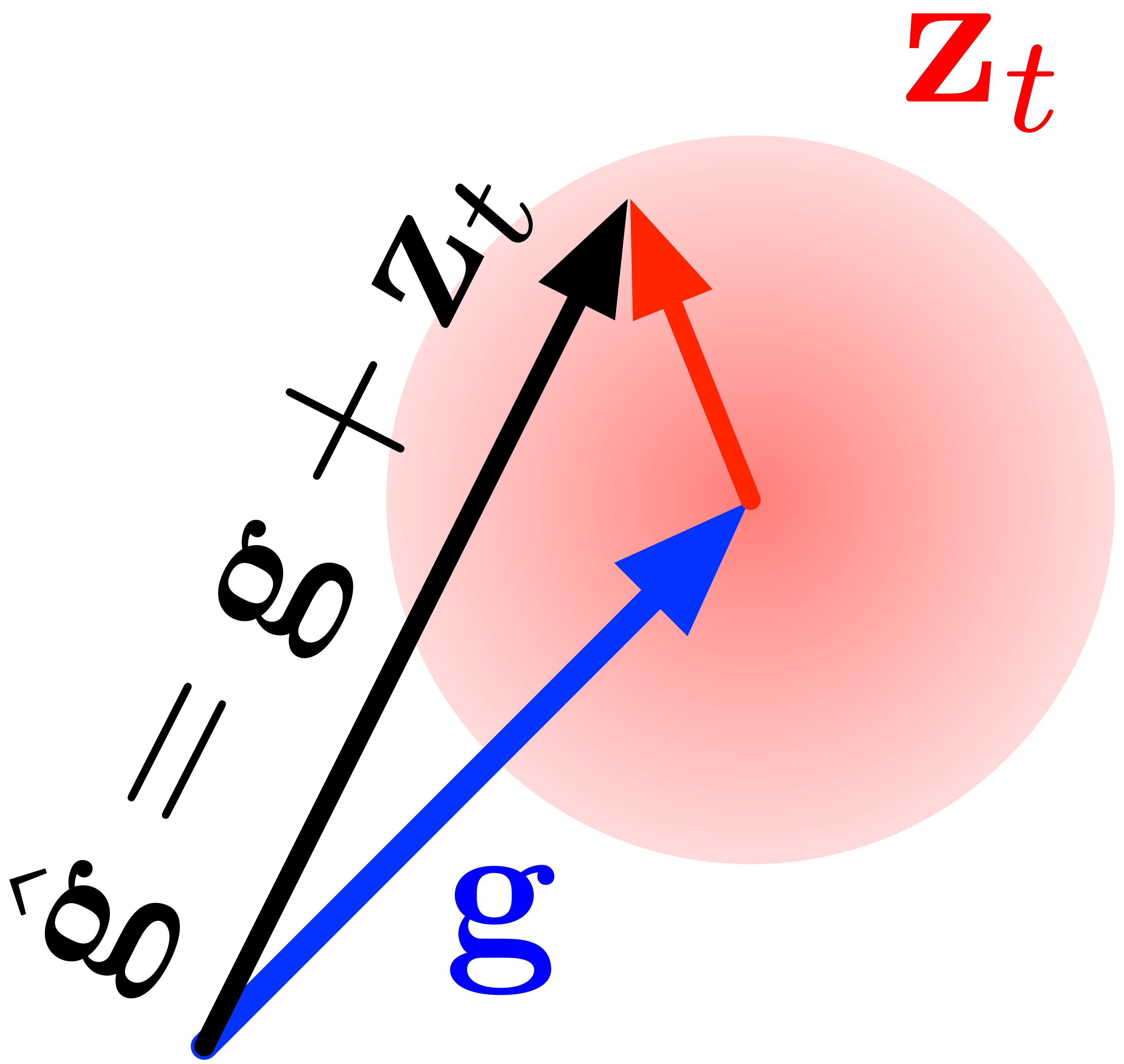
Deep Learning and DP

Privacy for neural networks



[Song et.al. 2013, Duchi et.al. 2014, Abadi et.al. 2016, Mironov 2017]

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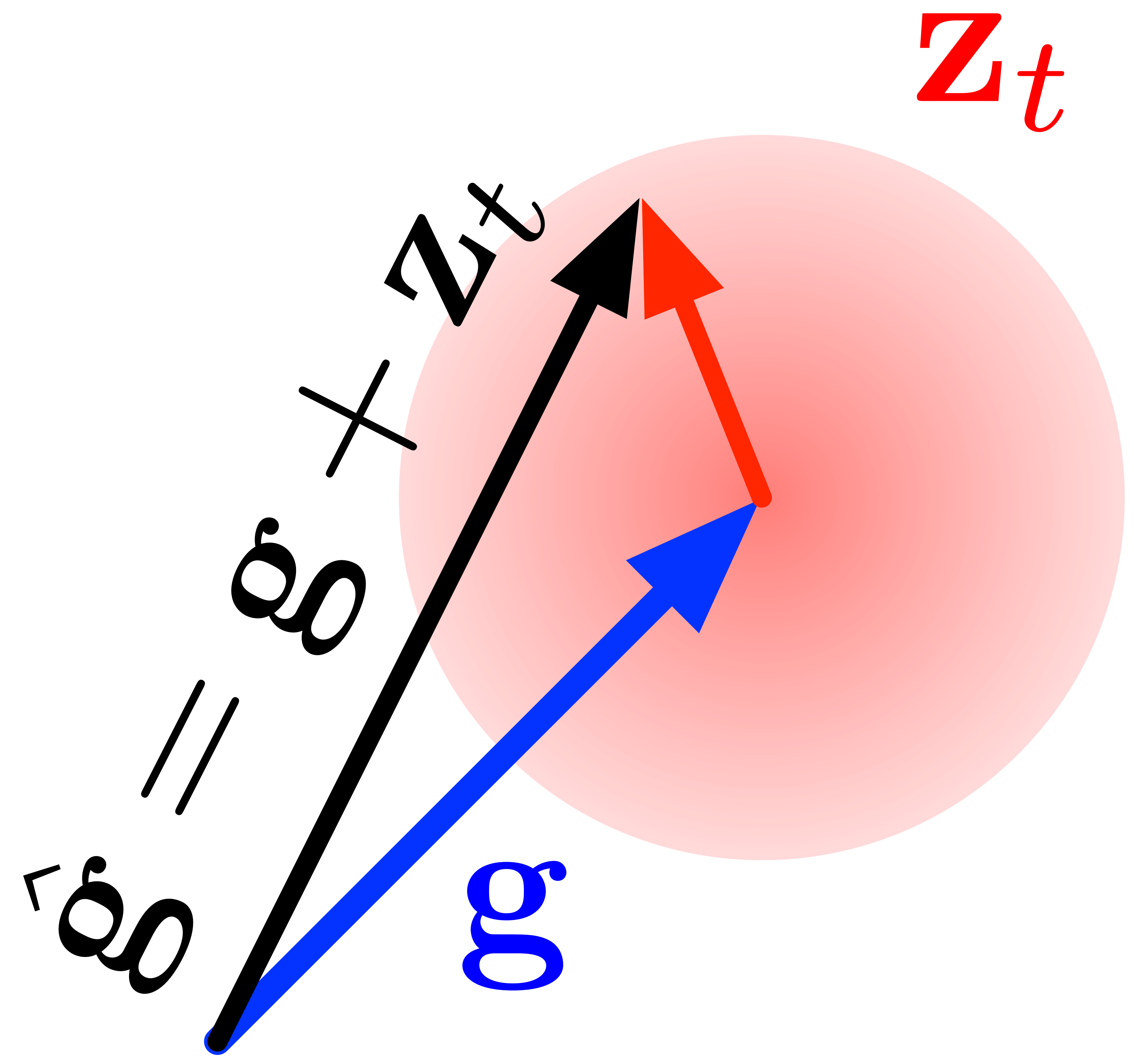

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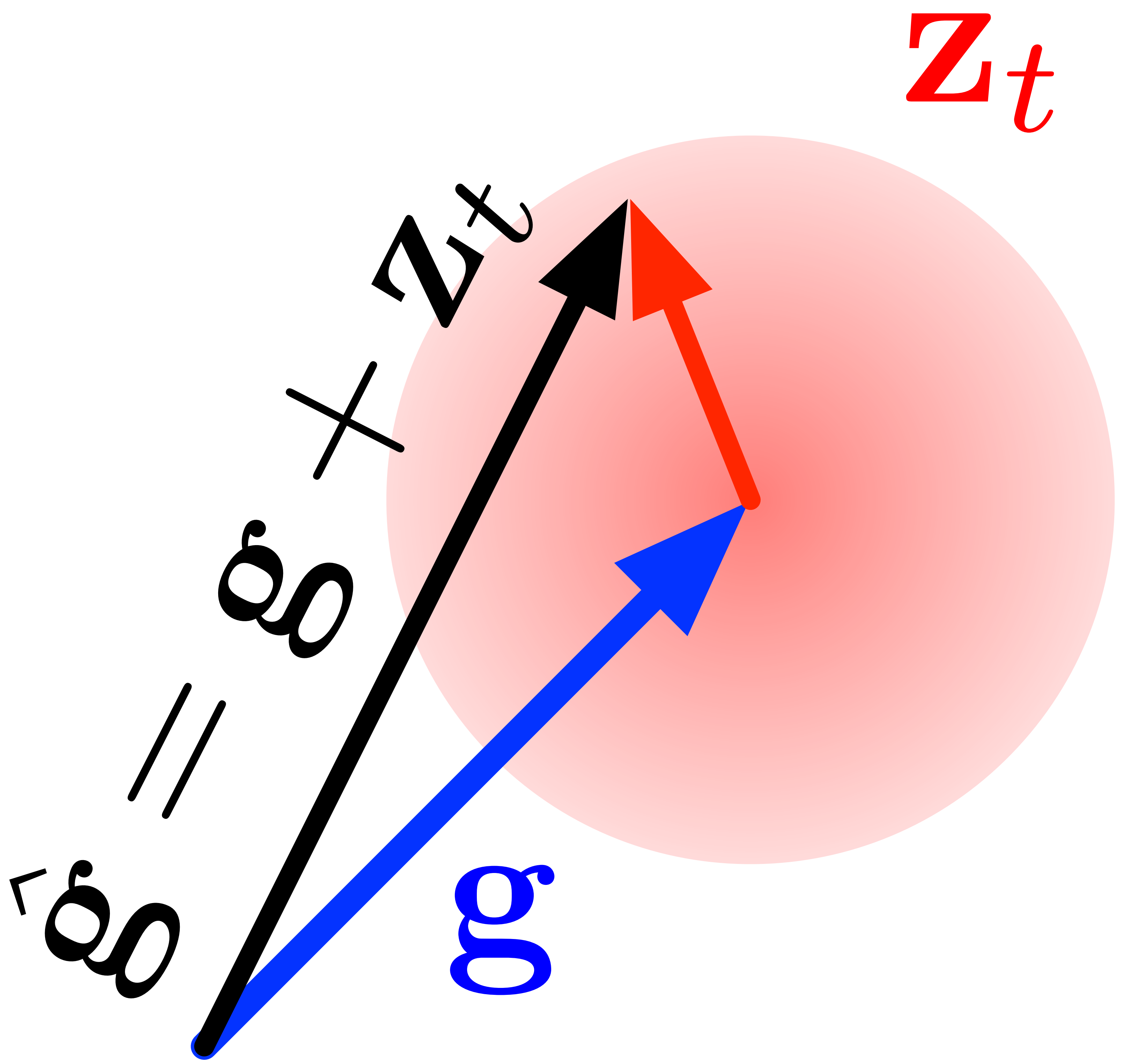

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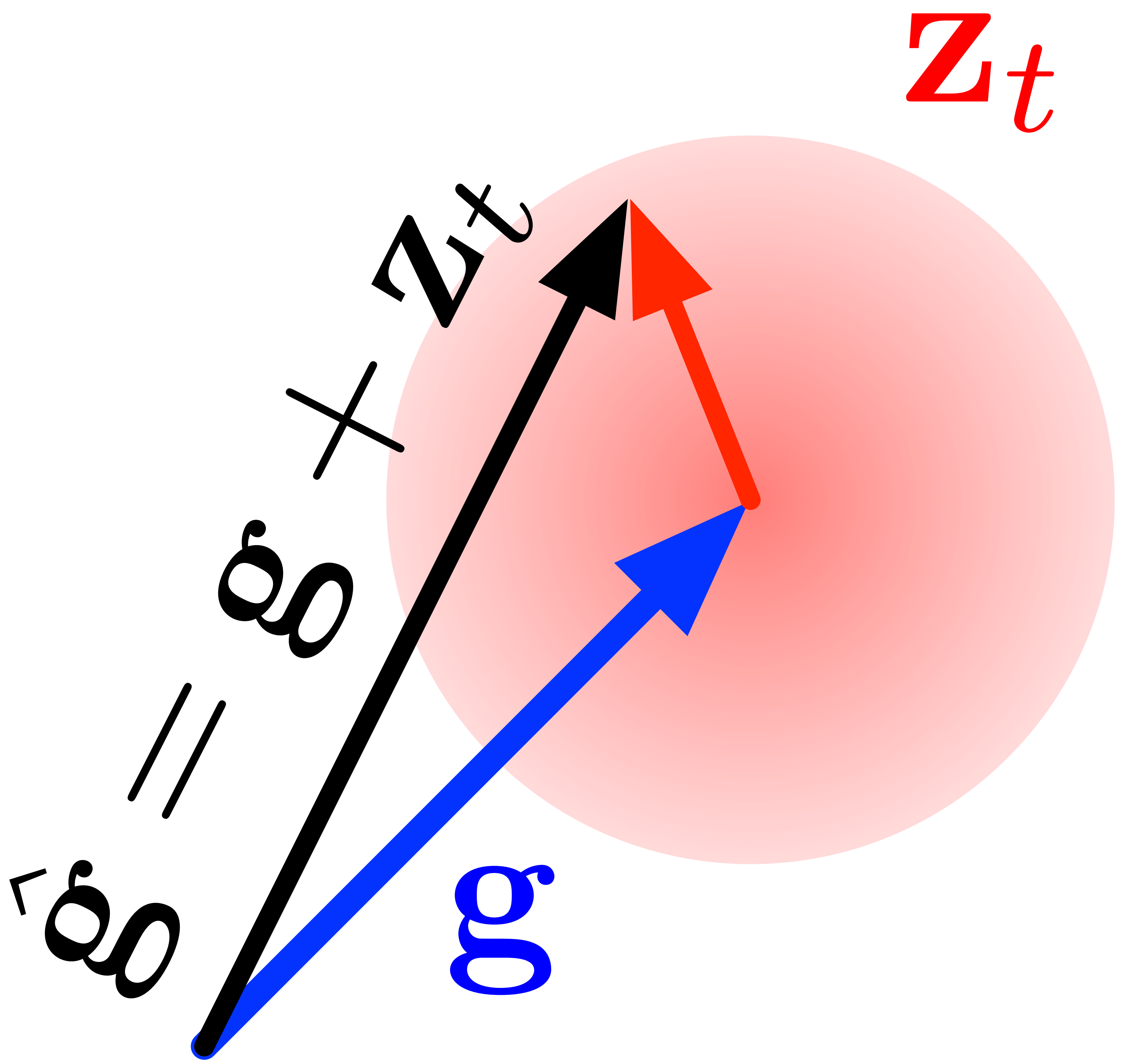

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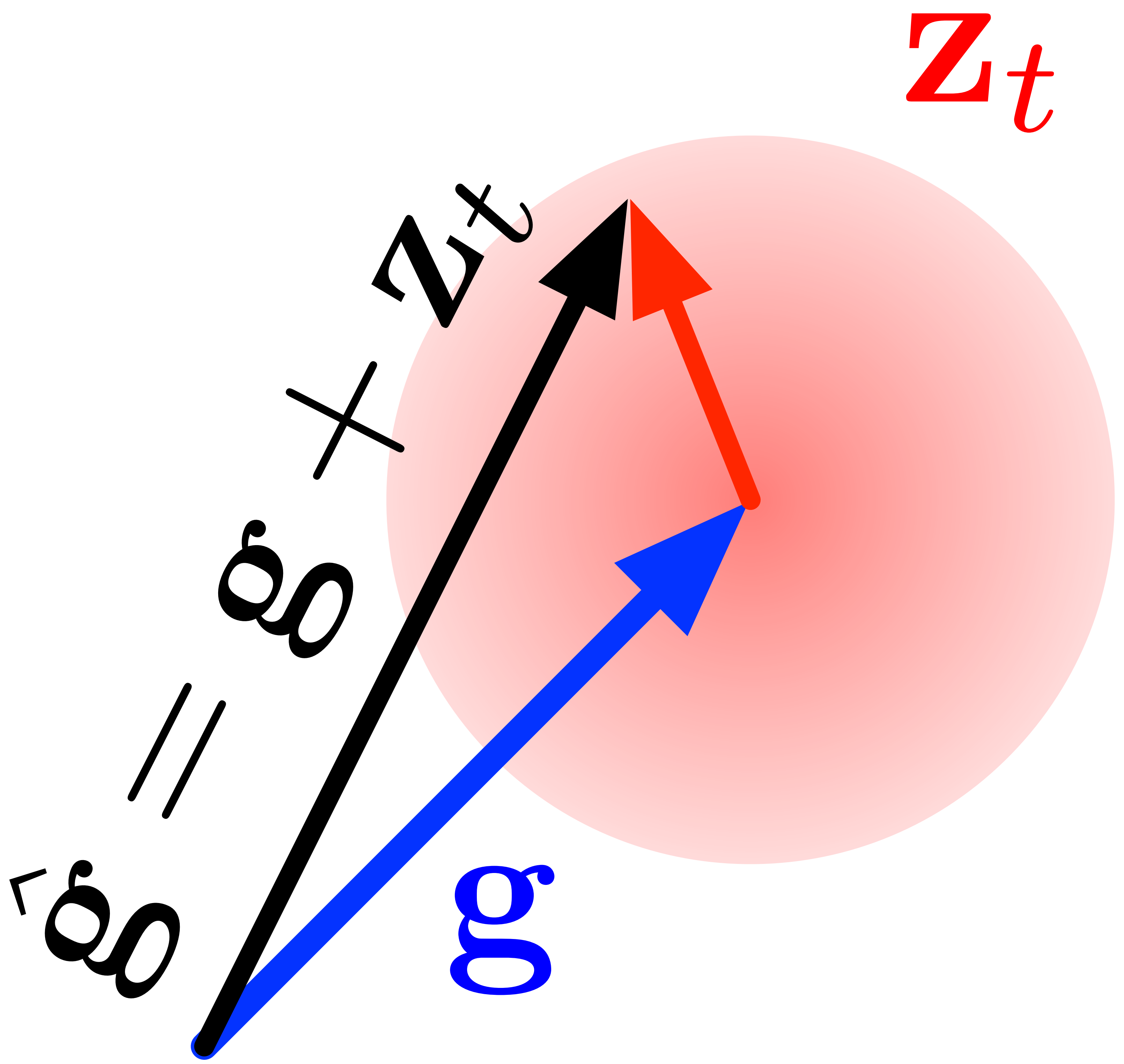

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Privacy accounting lets us track the overall privacy loss.

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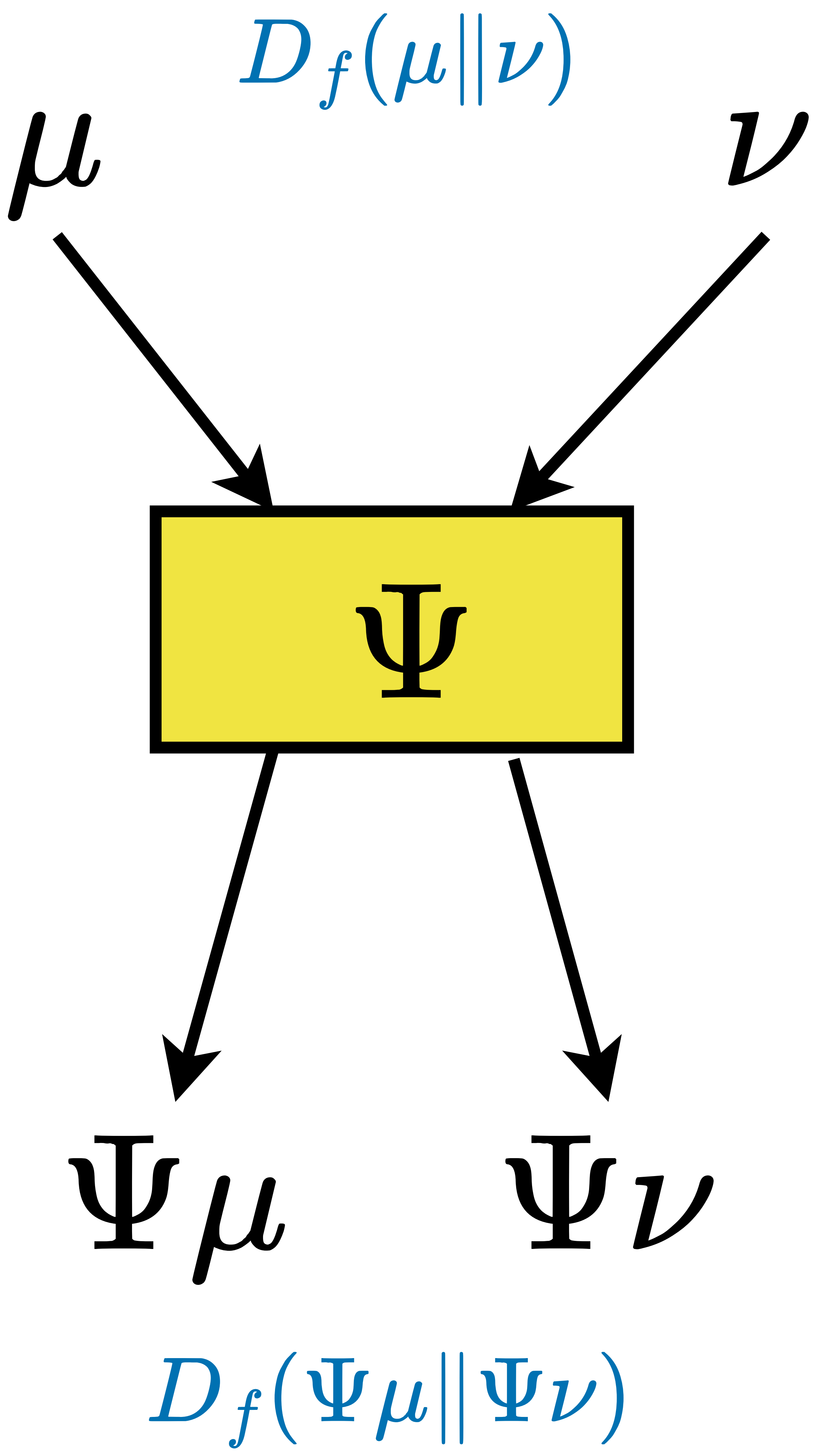
Strong data processing inequalities

Quantifying the privacy gain from post-processing



Dobrushin (1956), Ahlswede, Gács (1976)

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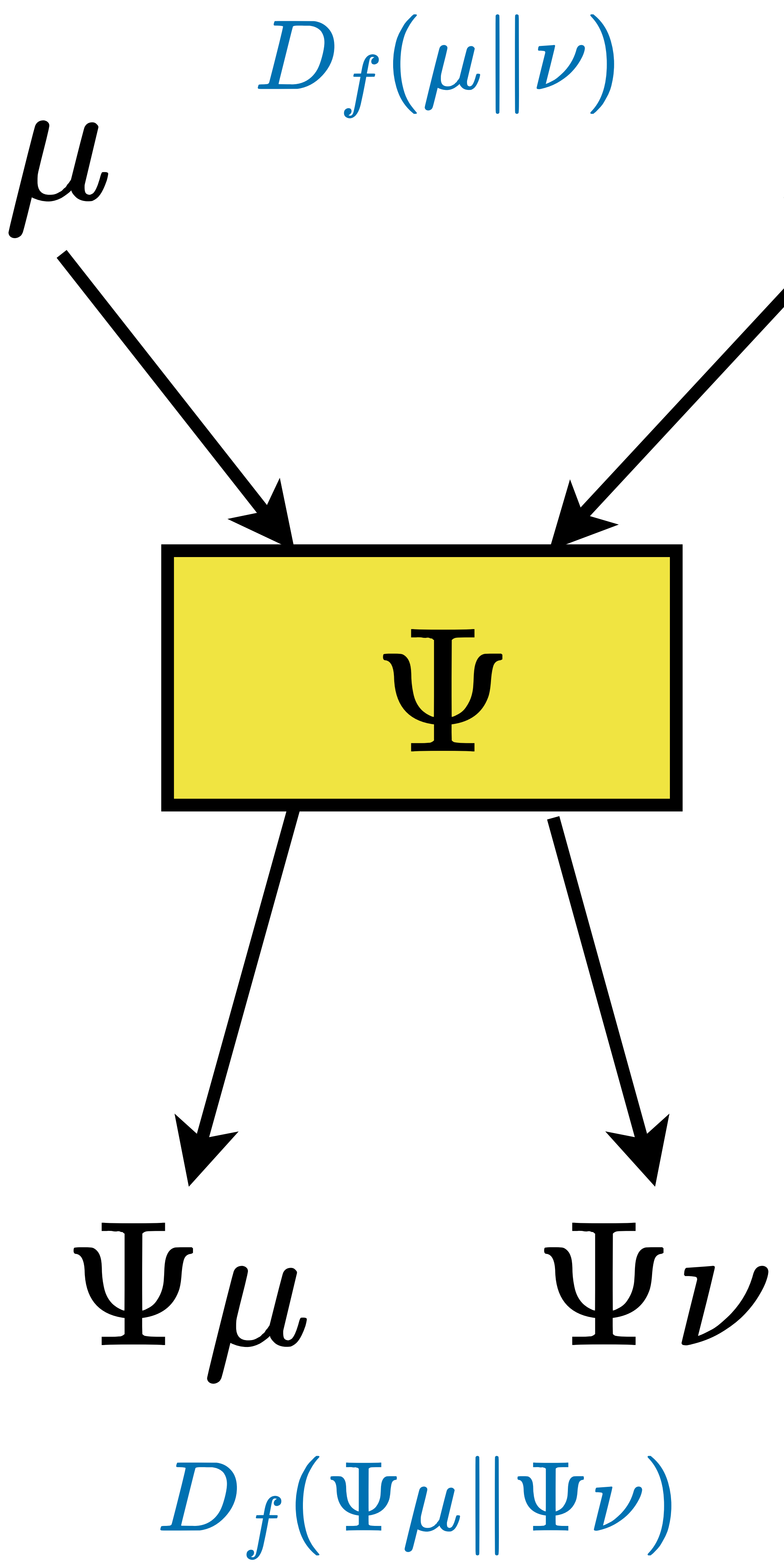

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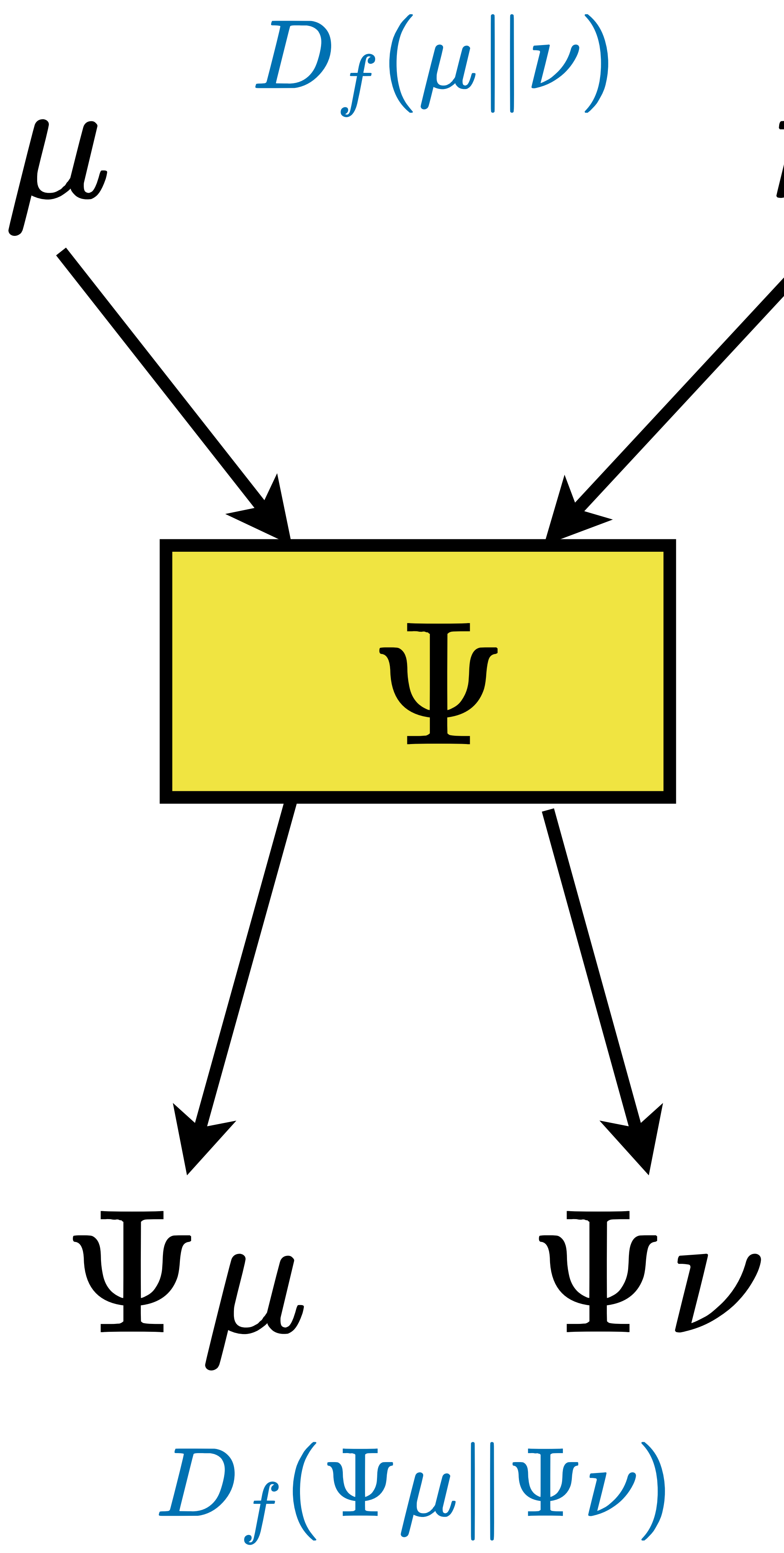

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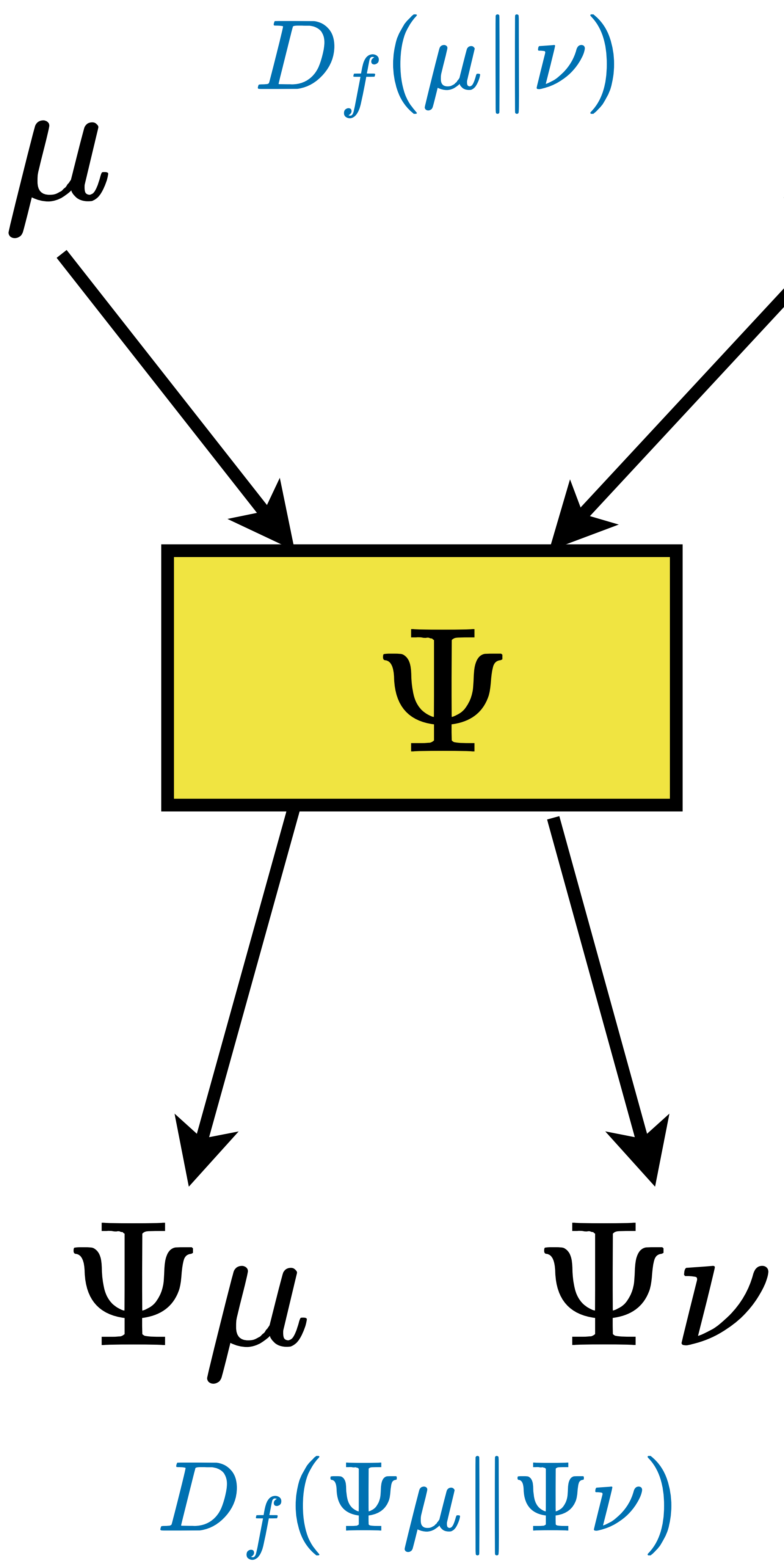

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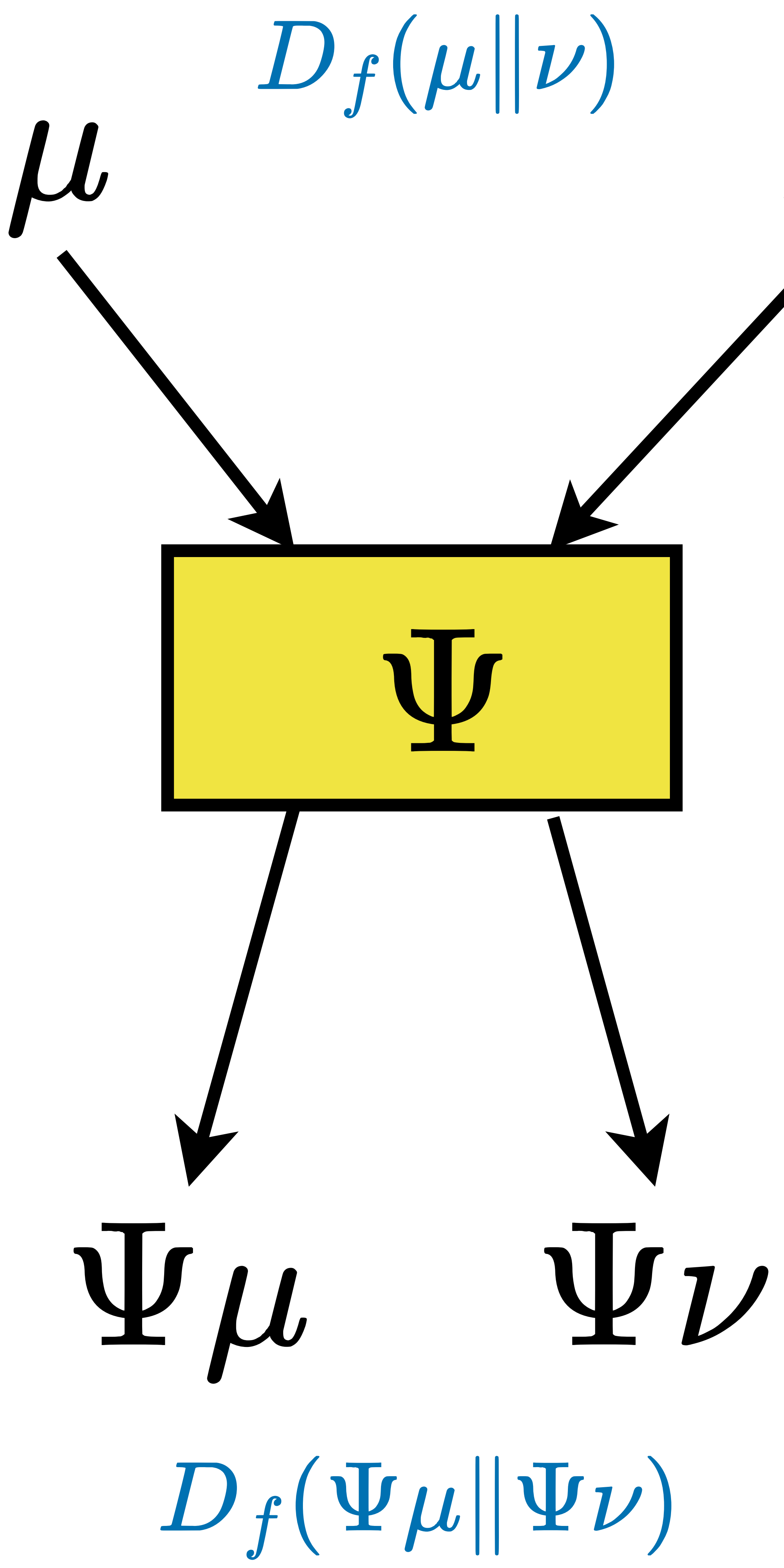

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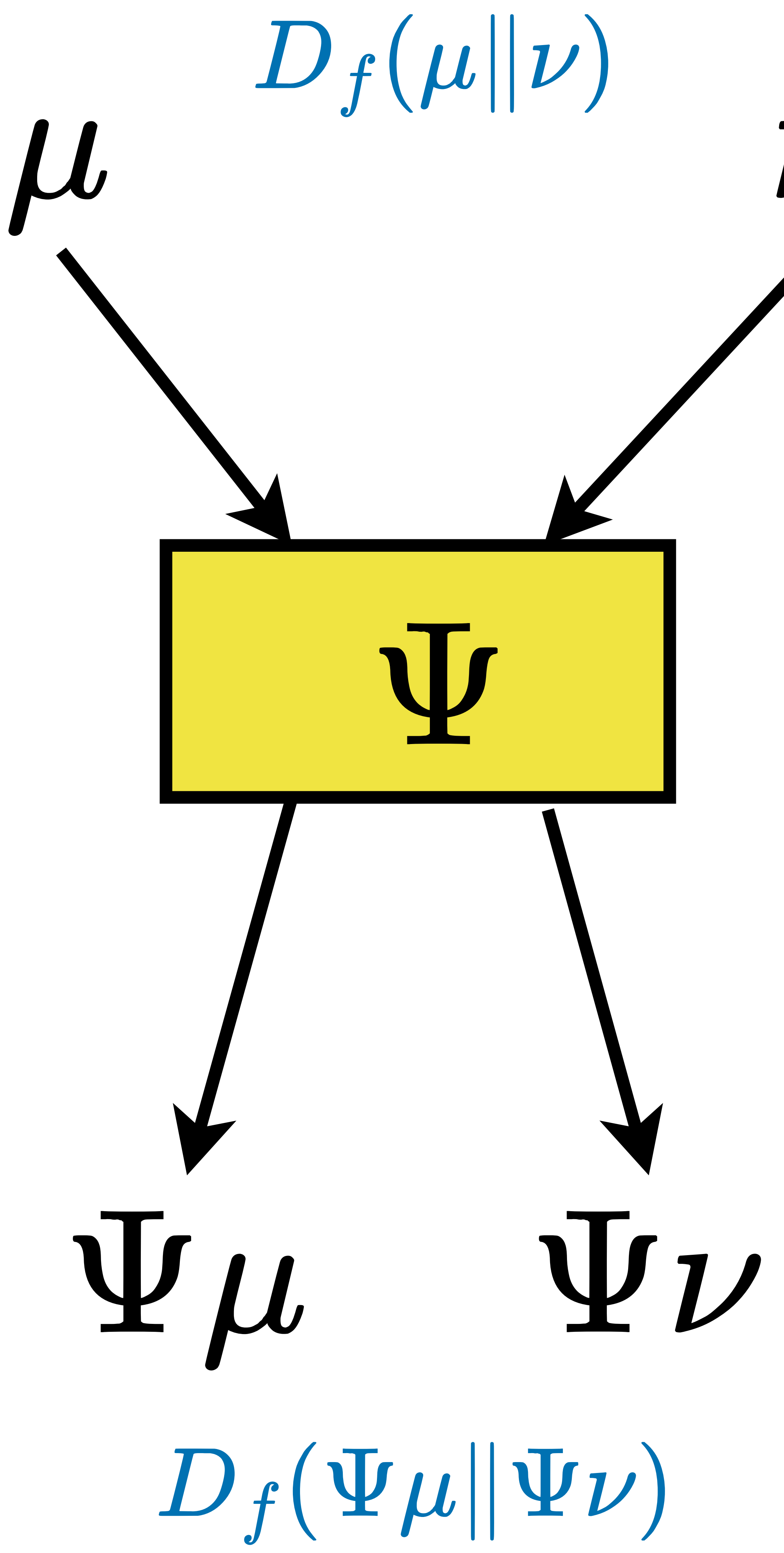

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If $\eta_f(\Psi) > 0$ this is a **strong data processing inquality (SDPI)**.

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Contraction for the 🏒 divergence

Applications to DP-SGD and LDP

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Application to noisy SGD

Iterations just tack on more Markov kernels

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Idea: analyze privacy for the *last* iterate by using the contraction for the E_γ divergence.

[Asoodeh, Diaz, Calmon (2020/2023), Asoodeh, Diaz (2024)]

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An abbreviated timeline

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Privacy amplification by iteration

An abbreviated timeline

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- Altschuler, Talwar (2022)/Altschuler, Bok, Talwar (2024) - projected, convex, bounded
- Asoodeh, Diaz (2024) - use data processing inequalities to remove convexity and smoothness assumptions for projected DP-SGD and regularized DP-SGD.

Contraction and Bayesian estimation

Focusing on the local model

[Asoodeh, Diaz, Calmon (2020/2023)]

Contraction and Bayesian estimation

Focusing on the local model

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> θ is a secret, the loss ℓ is a negative gain, and we look for the maximally leaky channel subject to an (ε, δ) constraint… (Is this right?)

[Asoodeh, Diaz, Calmon (2020/2023)]

Morning After a Snowfall at Koishikawa

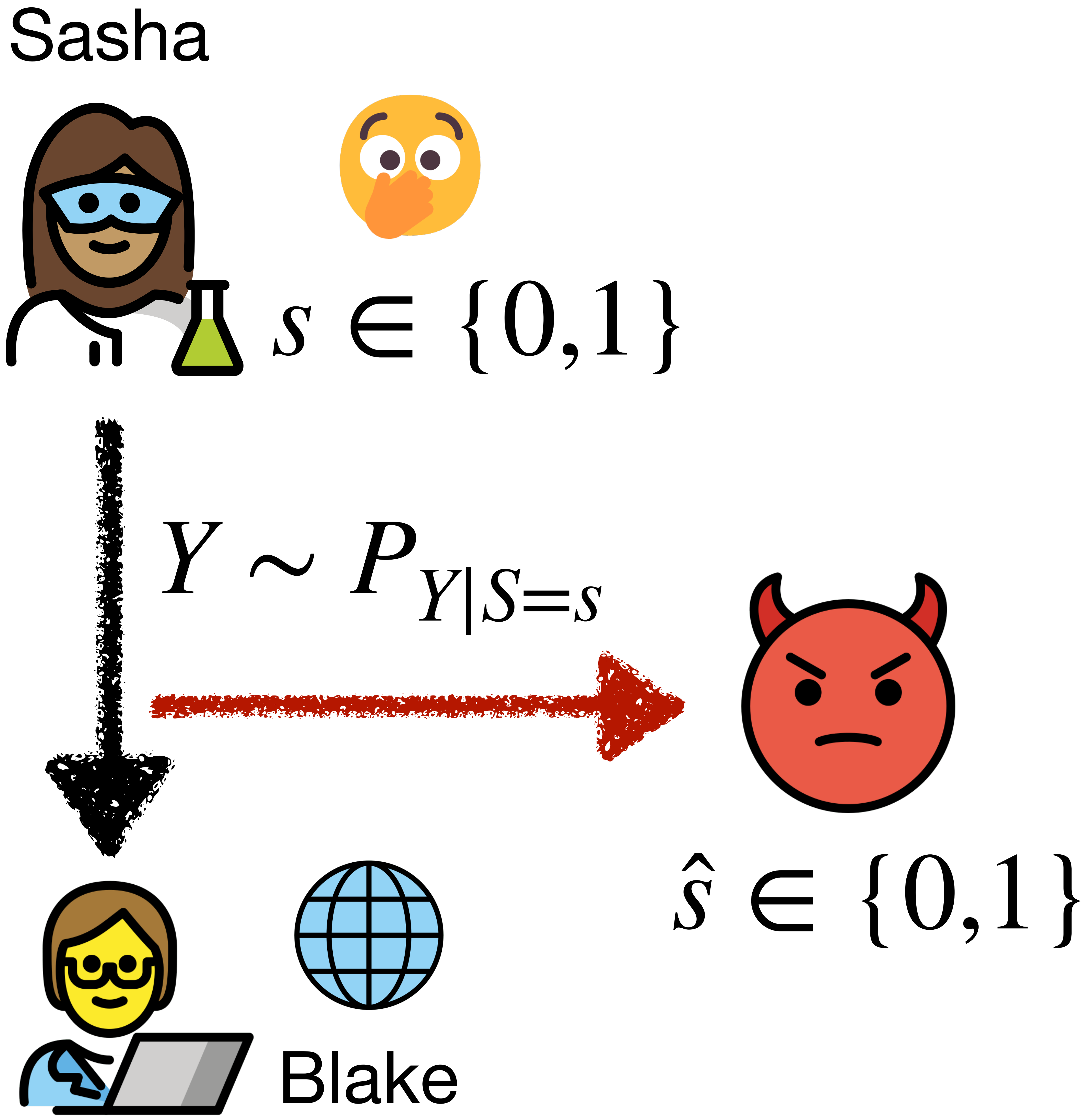
礫川雪の旦

Koishikawa yuki no ashita

other destinations

What we've seen so far

Let's start simple



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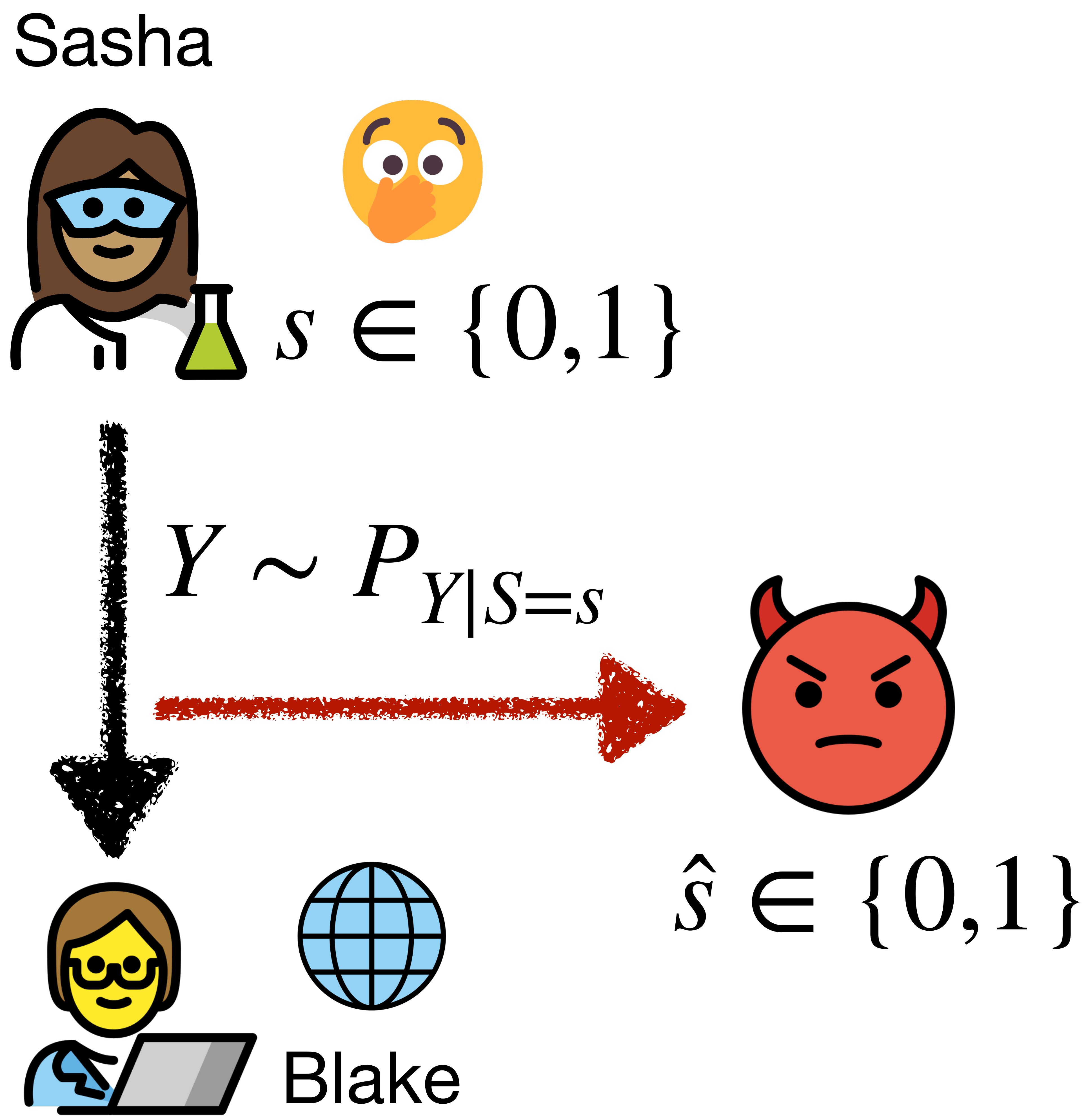

We started out with a simple story: protecting a single bit.

- Differential privacy both is and is not just as simple as hypothesis testing.
- Taking an information-theoretic view opens the door to better analyses.
- The gap between algorithms and analysis is shrinking.
- The gap between algorithms and applications is still large.

The gap between theory and practice

It’s wider than you might think

The gap between theory and practice

It’s wider than you might think

There are lots of issues with implementing differential privacy in practice:

- Approximate versus exact sampling (and side channels)
- Approximate versus exact optimization
- “Privacy amplification” and it’s implementation
- Numerical precision and floating points
- Managing privacy budgets


安野平兵衛
市川市紅

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maths

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computational stats

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engineering

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maths
computational stats
engineering
human-computer interaction

Several interesting challenges left for:

maths
computational stats
engineering
human-computer interaction
technology policy

The Great Wave off Kanagawa

神奈川沖浪裏

Kanagawa oki nami ura

Thank you!